The Mathematics Teacher

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Soviet science education and its challenge

NICHOLAS DEWITT, *Russian Research Center, Harvard University, Cambridge, Massachusetts.*

An address delivered at a conference sponsored by the Scientific Manpower Commission.[1] *Readers of* THE MATHEMATICS TEACHER *should read this because of its timeliness and comments on present-day policies.*

I AM DELIGHTED at this opportunity to speak to such a distinguished audience. The subject of my talk is communist education, particularly as it affects the training of engineers and scientists in Russia, and the issues and the challenge it creates for all of us. Every one of you have, by now, probably heard a great deal about this vexing subject of communist education.

Many things are being said on this topic today, but this is barely enough. I would like to remind you all that it was only a few years back that the vigor of communist education in certain fields was pointed out to us by a very few of our analysts and scholars. Hardly anyone paid much attention to their utterings—that in the long run it was not the present stock of resources, not the current advantages in the wealth and technology, but the brains and the minds which would decide the destiny of men. These brains and minds are largely developed by education, which transmits the values and the store of knowledge of the past, and which opens up the path of opportunity for the future. The voices of those who said that in this divided world the most precious resource is brain power were faintly heard.

As time went on, the harsh realities of the cold war forced some of our prominent public figures to realize, too, that education was one of the decisive factors. But the public was calm, uninterested, and utterly uninformed. Madison Avenue and the newspapermen still had not come to grips with this realization. Of course, you know, youngsters have to go to school just about everywhere, and the American public took it with all due calmness that Russian children most likely had to go to school, too. And the grasping of this simple fact did not require any newspaper headlines. One did not have to trace this out in scholarly books, in government files, or through congressional hearings. Little Ivan, just like little Johnnie, went to school—period. No story. No comment. And save for the few people in the entire country who kept a watchful eye on the unfolding drama of education under communism, no one gave a hoot about the fact that Ivan was learning not quite the same things in school that little Johnnie did.

But then, suddenly, the blast came. The newspapers got into the act. The display of modern weaponry, the jet fighters, the A-bomb, the H-bomb, the rockets developed by the Russians were much too much to be explained solely by Soviet reliance on captured "German brains" or "stolen secrets." The "challenge of Soviet education" and the "cold war of the class-

[1] Engineering and Scientific Manpower Education—Foundation of National Strength, Edgewater Beach Hotel, Chicago, Illinois, October 31–November 2, 1957.

rooms" were suddenly in the spotlight. The average American was bamboozled, yes, indeed, bamboozled, by these new issues.

The press headlines were almost to the point of hysteria, crying that Russian education does this and that and that we Americans should at once instigate crash programs of one type or another. Most of you will recall that these developments occurred two or three years ago.

But Mr. Average American could not believe that the educational effort of that "backward" Russia under savage communist mastery, unnoticed for over *three* decades, could be so significant and important as to shake him out of his smug complacency and everyday routine. The citizen sceptics were joined by educationalists, who felt that to admit the peculiar nature of Soviet educational gains would be suicidal to their prestige and the prevailing progressivist tenor in American educational practice. And they were joined by a chorus of pseudo-patriots whose theme song was "Anything they can do we can do better." To do this they added, "—and more, too, without a bit of extra effort." The mass-communication media began to beat a path of retreat, the politicians began to have some second thoughts, and the subject became not very "newsworthy," to say the least.

Instead of facing the issue of the Soviet educational challenge then and there in a calm and constructive way, the issue was dropped back into the public's lap. In a democracy like ours, it was said that "education is everybody's business"; but as it sometimes turns out, by the time each one takes care of his own business, "everybody's business" becomes "nobody's business." For over a year, then, until about a month ago, the issue of the "challenge of Soviet education" and all the unpleasant consequences it entailed was again banned almost to oblivion.

Three weeks ago, as you know, the cricket-like sound of the beeping satellite launched by Soviet scientists and technicians ripped like a thunderous tornado across the American political sky. It ripped to shreds some of the face-saving pooh-pooh's and attempts to hush-hush the significance of the issues which confront all of us. Again we find ourselves almost at the point of hysteria. There is a busy search for a scapegoat. Again there are cries for crash programs. There is no one to blame for what the Russians do save the Russians themselves, but every one of us is to blame, not so much for what we have not done as yet, but for our lack of understanding, of calm dispassionate understanding of the issues which confront us.

Among these issues, the central one is the realization of the fact that the Russians orient their educational efforts so as to maximize the returns from them for the advancement of political, military, and economic objectives. The communists do not believe in education for education's sake. They do not believe in education for the individual's sake. The Soviet state looks with contempt upon jacks-of-all-trades and masters-of-none. Its main objective is to offer *functional* education so as to train, to mould, to develop the skills, the professions, and the specialists required by their long-run development programs, who are capable of performing to the best of their ability the tasks of running the industrial and bureaucratic machinery of the communist state.

And in order to accomplish this, the Russians *were*, they *are*, and they *will* be training an army of scientists and technologists. Whatever we say about the numbers or quality, strengths or weaknesses, efficiency or waste, excellencies or failures of Soviet education, we must first and foremost recognize the fact of the *continuing* Soviet efforts to create and to run this vast new army of scientifically and technically competent, but politically, ethically, and morally ignorant soldiers of modern technology who are indeed serving, wittingly or unwittingly, the goal of communism. The real strength of this

army perhaps can be judged only in the long run. But we cannot for a moment discount its presence.

Newspaper headlines do not create issues, nor are the issues solved after they are out of the headlines. And so it goes for the issue of Soviet education. The second paramount issue which has to be faced squarely in talking about the Russian educational effort is that it is a *long-run proposition*. It takes fifteen to seventeen years just to educate a professional. It takes another ten years to make him an experienced engineer or an efficient researcher. The Russians recognized this fact three decades ago. The Soviet effort is a persistent drive. It is not an overnight feat. It is not just a passing fancy which, like an unpleasant rumor, can be stopped or silenced or forgotten at will. The Russians are not playing hide-and-seek. They are dead earnest in their ultimate objective that, if not today or tomorrow, then in a decade or a generation or two hence their absolute technological, economic, and military supremacy will triumph. Their educational program, they think, is aiding them in this goal. This is what they openly profess and preach, and they live and act accordingly. This *is* their goal. And anyone who does not realize this plain, hard fact is playing the fool. For folly it is, and I do not hesitate to use this word, unpleasant and harsh as it may sound, for I know no better one to describe the complacent wishful-thinkers who say this is not so, that there is nothing pressing for us to be concerned with now or in the long run.

These two central issues—first, the standing and growing Soviet army of professionals, technicians, engineers, and scientists, which is deployed not only domestically but increasingly to underdeveloped countries of the world in order to further Soviet political ambitions through technical aid, and second, the long-run objective for which this army is being mobilized—contain a host of subsidiary issues and problems. Among these are the issues of numbers, of quality, of motivations, and performance—and I would like to mention these in more detail.

As to numbers, the first thing we have to know is the size of this army. This is the so-called "issue of numbers," which has time and again been dismissed or minimized by some observers. Let me first sum up briefly the numbers themselves and then turn to their implications.

As of 1956, the Soviet Union had an estimated 20 million persons with completed secondary nonspecialized education. In addition, at the beginning of this year, 1957, the Soviet Union employed in its civilian labor force (excluding the military, that is) some 2,630,000 professional specialists with completed *higher* education, and about 3,600,000 subprofessionals and technicians with specialized secondary education. Among the higher education graduates employed in the Soviet economy there were about 720,000 engineers; 160,000 agricultural field specialists; 330,000 medical doctors, and some 500,000 science and mathematics teachers and university-trained scientists in non-applied fields. Altogether, there were about 1,700,000 people in scientific and engineering fields, which represents almost two-thirds of all Soviet higher education graduates. And I wish to remind you that although we had better than twice as many college graduates, in the respective categories of science and engineering in the past year the United States employed a slightly smaller number than did the Soviet Union.

The work of these professionals was aided by over 3.6 million supporting personnel, specifically trained for this purpose. In industrial fields, for example, professional engineers were supported by 1,000,000 technicians; medical doctors were aided by 900,000 nurses and other medical-field supporting personnel; 300,000 subprofessionals were in agricultural fields. I wish we could compare this with the United States; but as you may know, even after long years of talking about how

important technicians and supporting personnel are in our production effort we still don't know how many there are. Even without making a detailed comparison, the Soviet professional manpower figures look impressive enough, particularly if we consider the fact that these 1957 totals reveal eight- to ten-fold increases as compared with those three decades earlier. This army emerged in just about the course of *one* generation, and, at that, a generation which lived through famines, political purges, and devastating war.

To complete the numerical picture, we have to consider the current flow, the new additions to this stock. This year the Soviet general education ten-year schools graduated about 1,500,000 youths, and of these about two-fifths continued their educations further. Subprofessional specialized schools graduated about 500,000, among whom almost 180,000 were technicians and engineering aids for Soviet industry. Institutions of higher learning and universities graduated almost 280,000 of whom 70,000 were professional engineers and about 80,000 were in other scientific fields (biological, medical, and agricultural fields, science teachers and university-trained researchers). While the totals of higher education graduates were roughly comparable in the two countries, in scientific and engineering fields Soviet graduates outnumbered our annual additions better than two to one. Such a Soviet lead has been in evidence for the last six years, and there is no evidence that it will subside in the future. This year enrollment in Soviet engineering schools edged up to 500,000. Thus, to sum up, the stock of trained men in science and engineering for the two countries was about the same, but the Russians have a two-to-one edge in the new flow.

Now let me turn to the "issue of numbers" as such. First, it would obviously be silly even to mention these statistics if they did not reflect a rough comparability of training. For the time being, let us say that in the over-all judgment of most experts, Soviet higher education graduates particularly in engineering and scientific fields, are not any worse prepared than our own. This is a workable and provable assumption, on the basis of which the numbers for the two countries can be compared. But this just sets the stage for admitting the issue.

The first implication is that our modern science and technology are so complex that specialization is inevitable. There is a matter of disagreement as to what degree this specialization is to be carried out without detriment, but that it exists and will continue to exist is beyond dispute. In turn, individual specialization calls for teamwork and integrated collective effort and creates many organizational problems. All this greatly increases the requirements for engineers, scientists, and supporting personnel in terms of numbers alone. It is obvious that the larger the pool, the better the chance of efficiently fitting an individual with suitable qualifications into a given research or engineering task, a team, or a project.

The performance of everyday research and development nowadays requires numbers. Furthermore, though we may be scornful of Russian mobilization and priority allocation programs in some scientific and engineering development fields as being at times inefficient and wasteful, the plain fact is that if there are no numbers, there is nothing to mobilize. Numbers not only add to the mobilization potential, but mobility and flexibility in everyday research effort are dependent upon them.

The third point is that extraordinary talent in almost anything is randomly distributed. It is a matter of probability that large numbers increase the chances of picking up truly extraordinary persons with outstanding ability for more creative tasks in science and engineering and for pioneering and discovery work on the frontiers of knowledge. Were it not for these three implications, most of us, I am sure, would agree that numbers alone would not matter much.

This brings us to the second problem, namely, the quality of those who enter this standing army. It has already been stated that Soviet higher education graduates are certainly not any worse prepared than our own. Those who have looked into the perplexing muddle called "quality" of education in the two countries are willing to go one step further and say that in the engineering and science fields the training of Soviet graduates is probably of *better* quality than that of our own graduates in these fields. Why? Let me mention several factors which I think are significant.

The Soviet Union runs a very peculiar educational system. In common with the American tradition of public education, the Soviet system is said to be designed for so-called mass education. This is particularly true of the Russian ten-year general education school, the counterpart of our twelve-year system of primary and secondary education. Some 28 million children attend these schools in Russia today. The point is, however, that of the 9,500 instruction hours given during the ten years in these schools, about half are devoted to sciences and mathematics. This is part and parcel of "general education"—Russian style. The curriculum is standard for the country as a whole, and there are no electives. Every pupil *has* to take five years of physics, six of mathematics, three of biology, four of chemistry, six of foreign languages, and so on. Either the youngster succeeds in passing the required subjects or, if he fails, he is out, and is absorbed by an elaborate system of trade and vocational schools.

The standard curriculum and the selective process of the ten-year schools are those in common with European educational practice. This year (1957) about 1,500,000 were graduated from Soviet ten-year schools. This means that only about one-third of those who entered school ten years earlier succeeded in graduating. But note that each one of those who *did* graduate had, roughly speaking (in terms of hours devoted to science subjects), *five times more* than the minimum stipulated for entrance to a school the caliber of the Massachusetts Institute of Technology. Examination of Soviet syllabi and textbooks in physics and chemistry in the last two grades (nine and ten) of the Soviet secondary school reveals that they are comparable with those used in introductory courses in these subjects in our colleges. Thus, it is the *secondary schooling* in the Soviet case which accounts for early and vigorous exposure to the sciences and makes for better student material from which to choose for further training.

But choice and selection do not end there. In order to make the transition from secondary education to a higher educational establishment, the student is subjected to two screening processes. At the end of the ten-year school, he is subjected to matriculation examinations in all of the subjects covered in the secondary school, and good grades are a prerequisite even to be considered as an applicant for further training. The second screening process is in the hands of the higher educational establishments themselves and takes the form of a number of competitive entrance examinations, ranging anywhere between three and six in number. In engineering and scientific fields, for example, these examinations usually cover five subjects: physics, chemistry, mathematics, the Russian language, and a foreign language. It is on the basis of these competitive examinations that actual acceptances are made. In addition, students are subjected to more or less effective political screening. As a result of this selection process, last year only one out of more than seven applicants was admitted to a higher educational establishment. In engineering and scientific fields, selection is even more stringent, and some better engineering schools and science divisions of universities are known to accept only one out of twelve applicants. As a result of the competition which they must face to be admitted, students who do enter higher

educational establishments, are, for the most part, expected to be of unquestioned ability.

Training in scientific and engineering fields in higher education is divided among numerous specialized institutions, the training in which last usually five to six years. These institutions, unlike those in this country, are grouped along functional lines. A large proportion of future research scientists are trained in the science divisions of universities. Secondary school science teachers are trained in pedagogical institutes that strongly emphasize subject matter rather than methodology and teaching practices. Future engineers, agronomists, veterinarians, medical doctors—all are trained in a *separate* specialized higher educational institution which, from the third year on, gives them oriented professional training. The student has a choice as to which institution or which specialty field he wants to enter, but once he enters that specialty, he has little chance of changing his field.

The length of the engineering course in higher education or of a science program in a division of a university is about five and a half years, during which time the student receives about 5,200 to 5,500 instruction hours in 35 to 40 individual subjects. This compares favorably with the 3,700 to 4,000 instruction hours and 22 to 25 subjects which are common in U. S. practice. The work load of students is extremely heavy, and ranges from 35 to 40 instruction hours per week and from 25 to 50 hours in assignments, individual study, and required reading. During his academic history the student takes about 40 examinations required for the diploma, usually attends three industrial practice or practical work sessions, and is required to prepare and defend a thesis dissertation. Under these conditions, out of 100 students who enter the higher education, only about 75 succeed in graduating. After receiving such training, the graduate is assigned by the state to a position which he must keep for at least three years.

As far as the amount of study material, as far as basic textbooks, as far as curricula and syllabi are concerned, their examination and comparison with our own suggest that a Soviet student in an engineering program or in a scientific field at a university should graduate with qualifications not inferior to those he would have had he attended a good engineering school or university science program in the United States. At times, they may be slightly higher.

There are two differences, however, The Soviet graduate engineer, in addition to the basic knowledge of scientific fundamentals and engineering tools, will have a package of narrow specialty tools or applied technology fundamentals, for about 22 to 28 per cent of the instruction time is devoted to specialized subjects. This is done in the last two years of study and to a great extent resembles on-the-job training programs offered by our large manufacturing firms for newly-hired engineers. Furthermore, every student is required to spend six to eight per cent of his time on strictly political subjects and indoctrination courses.

The rigorous selection process and the academic demands made upon the student counterbalance, in a sense, his incentives and motivations. And the ultimate question arises, why is higher education, particularly in the sciences and in engineering, such a desirable goal to the Soviet citizen? What motivates him to seek it?

We have to mention first that higher education students get paid for going to school. The Soviet state controls all types of education, and all educational expenditures are borne by the state, which spends annually therefor about seven per cent of the Soviet gross national product. Education in the Soviet Union is largely tuition-free. If this is not enough of an incentive to stay in school, there is even an excellent chance of getting paid, for about 80 per cent of Soviet students in specialized schools, vocational training, higher education, or post-higher education get a

monthly living allowance stipend. This stipend ranges in *higher* education from one-third to two-thirds of the prevailing monthly wage, up to an amount substantially higher than the prevailing average wage of Soviet salaried workers and employees, for a post-higher education student. These stipends are invariably differentiated in favor of science and engineering students. Thus, in effect, a successful student gets *paid* for going to school. It is not a burden or a sacrifice; it is an advantage.

These motivations are reinforced by external pressures and incentives. One of the most significant external factors has to do with psychology and opportunities. It is a peculiar kind of psychological pressure. One recent visitor to the Soviet Union has remarked that he could not discover there "a true Marxist, a believing communist, and an honest man at the same time." This is a bit perplexing but most to-the-point observation.

Marxism claims to be a science. Communist propaganda extols the objective, cognitive powers of science. It acclaims the supremacy of science in productive processes and the organization of society. In short, Science with a capital "S" is on permanent display in the dogma. To this is added the traditional Russian intellectual attitude toward knowledge as being omnipotent and "good," no matter for what; and we get what almost amounts to a mania on a national scale for science and knowledge.

In politics, in social problems, in subjective expressions of mind and intellect, a Soviet man really *has to be utterly dishonest* with himself in order to be a Marxist and a communist. It is dangerous to be true to himself. It is not so in science. A scientist or an engineer can seek fulfillment of his lifework aspirations on the very grounds of the doctrine which he may question and doubt. Time and again we have observed Soviet scientists and engineers separating objective science from ideology and politics. Time and again we witness party theologians trying to impose and enforce unity. In the Soviet setting, the creatively inclined individual is most likely to seek escape, yes, indeed, escape, from political and social reality, from the theology of communism, in a scientific or engineering career.

It is a paradox, a tragic dilemma of Soviet science education that it is propelled by psychological forces largely not of its own making. Many good men who flock to its ranks are there to escape the hazards of political reality, to seek a safer haven, and to preserve the tiny bits of human honesty and dignity largely destroyed or shattered in the hypocrisy of communist tyranny.

Complementary to these psychological motivations, if we may call them that, are material incentives. They are generous, indeed very generous, compared with the drabness, shabbiness, and deprivations of the average Soviet worker or peasant. It is quite common for engineering salaries to be *three* times higher, professional salaries to be *five* times higher, research scientist salaries to be *seven* times higher, and the salaries of outstanding designers and academicians to be *ten* and more times higher than those of the average wage earner. Prizes, honors, medals, scarce new apartments, suburban homes, private cars, and scores of tangible favors and privileges—all are bestowed by the Soviet state upon those who show that they are good performers and producers of products, of research, of ideas which can directly or indirectly enhance industrial and military power or score a political or propaganda advantage.

While the rewards are high and, on a sliding scale, are much more generous than those we offer in our own system, the penalties are just as harsh. Those who cannot carry the burden of directed research are doomed to obscurity, to deprivation, or even worse. Do you know that during the biggest and bloodiest purge conducted by Stalin and his heirs who are still today in the Kremlin, the purge of

1937, more Soviet professionals perished than during the entire period of World War Two? Whoever heard of a Russian scientist refusing to work on a top-secret project for the military? Whoever heard of a Russian engineer announcing his invention in a trade journal before it was cleared by the plant, by the ministry, by the planning board, by the military, and finally by the censor?

Soviet science *does* serve the state, the military, political, and power objectives. But in serving these objectives it often fulfills scientific purposiveness and thus gets full benefit and full credit. And this is what we often fail to grasp in trying to understand the Soviet effort. We repeatedly and persistently underrate the Soviet potential in science and technology. We forget that the Russians have reached such a position with their scientific army that they can achieve any research or engineering objective that we can, but they cannot achieve all of them at once. And the scale of priorities dictated by the planners is very much in evidence. At times these priorities and political dictates result in a genuine scientific accomplishment, and at other times lead to obvious "scientific" nonsense. The army has in its ranks brilliant soldiers as well as outstanding political charlatans and scientific clowns. Too often we pass judgment without looking at both sides of the coin.

This, then, raises the last problem which I would like to mention. There are two ways in which the price of scientific and technological progress can be measured. One is the way in which science and engineering do things for us, for our general welfare, our well-being, our standard of living, if you will. The other is what they deprive us of, what cost and sacrifice we have to bear in reaching a certain goal. And the point is that the Russians, willingly or not, are paying the price of sacrifice.

The greatest lesson we should learn from such a feat as the launching of the Soviet satellite is that leadership is an expensive proposition. If we are determined to retain our technological and scientific leadership, then we cannot afford to be the loser too often. We should at last embark upon a realistic program and determined deeds in fostering more and better science education for all our youth, much more generous support of basic science and research, and in increasing the social prestige and material incentives for our individual teachers, scientists, and engineers. We have a very difficult task ahead. In each encounter we will have to match our men of science, one by one, against this Soviet army. It is the skill and virtue of each man that we will have to be dependent upon. For we must remember that a democracy cannot survive without the best in education for each one, but that a lot of education neither makes nor is the safeguard of democracy.

Thank you, very much indeed.

To help develop insight into the division process (or just for fun), when your class is studying synthetic division suggest that they try reversing the process to develop a "synthetic multiplication" for multiplying a series of linear binomial factors.—*R. Saettler* and *K. Lake*, *Bradley University, Peoria, Illinois.*

Some implications of twentieth century mathematics for high schools[1]

ROBERT E. K. ROURKE, *Kent School, Kent, Connecticut.*

Recent developments in mathematics have a bearing on the mathematics taught in the secondary schools. This article shows how these developments should affect elementary algebra.

MATHEMATICS IS ON THE MARCH. One does not need to be a seventh son of a seventh son to see, abroad in the land, an unprecedented interest in mathematics, a new and better appreciation of its potential contributions, and a great and growing discontent with the content and presentation of the mathematics program from the first grade right through college. Newspapers, magazines, and professional publications scream the message: "The nation badly needs more and better-trained scientists." New discoveries at the frontiers of research and new applications call for new goals, new curricula, new texts, and new methods of teacher training.

And I mean "new"—not just a permuting of chapters in textbooks, not just a reprint of the old stuff in technicolor. The technical needs of our society demand a radically different type of preparation for the physical sciences, the social sciences, business management, and national defense. One economist has said that "the relation between mathematics and economics may someday resemble that between mathematics and physics." The handwriting is not only on the wall; it is all over the building.

[1] A paper presented at the annual meeting of the National Council of Teachers of Mathematics in Philadelphia, 1957.

Of the roadblocks that stand in the way of a really new curriculum, I shall mention two: the difficulties of semantics, and the resistance offered by the defenders of "the old time religion." The first can be overcome by care and patience, the second by exhibiting the facts.

If builders of a new curriculum fail to spell out carefully what they mean, they must expect misinformed comments about a new program. If they urge a greater emphasis on ideas and concepts, their hearers may repeat: "These fellows are going to abandon all manipulation." If curriculum planners vaguely suggest that some notions about sets should be introduced in high school, they are likely to get the reaction: "Boolean algebra is to be taught in the ninth grade!" After one rather nebulous presentation to a group of teachers, I overheard a teacher comment: "I'd like them to try teaching symbolic logic and theory of sets to that bunch of mine."

To avoid such semantic difficulties, the suggested approach to new topics and new concepts must be set forth in detail, tried out in the classroom, experimentally, and made available in polished form to teachers and textbook writers. The Commission on Mathematics of the College Entrance Examination Board is proceeding along

these lines.[2] Later in this paper, I shall attempt to spell out a few ideas about the role of sets in high school algebra.

The second roadblock leads to comments such as "What is wrong with the traditional program?" These comments often come from sincere believers in the traditional program; it is wrong to assume that all such questions come from rationalizers of lethargy or defenders of incompetence. There is nothing "wrong" with the traditional program; it is simply out of date. And since our time is finite, we must replace topics once important with others now considered of greater importance. For example, one reads in *Items*, published by the Social Science Research Council, that "recent developments in various fields of social science have created demands for mathematical training that are not satisfied by traditional curricula." Many others have struck the same note. One need only point to the ever-growing literature attesting to the fantastic development of mathematics in recent times, and to its many new uses.

Now, in my opinion, this does not mean that all traditional topics must be discarded—lock, stock, and barrel. On the contrary, it seems to many of those interested in creating a new high school curriculum in mathematics that much of the old is not deadwood, but livewood; or, rather, it can be enlivened by a new point of view. I offer what I believe to be five good, sound reasons for using certain ideas of modern mathematics in building a high school curriculum:

1. *To clarify.* We have a lot of fuzz in our texts and in our teaching. At many points, modern thinking can help us to sharpen the ideas and bring them into better focus.
2. *To simplify.* We need to delete many useless and vague terms. Modern notions can help us trim off a lot of the fat.
3. *To unify.* We have long talked about the need for integration and integrating concepts. Where can we find a better integrator than the concept of a set? Sets give a new unity to the study of equations, inequalities, relations, functions, sample spaces, and so forth.
4. *To broaden old ideas.* We can gain and give new insights into many topics, such as, for example, graphs and loci, thus developing new interest and enthusiasm.
5. *To introduce important new ideas.* A great portion of the modern vocabulary and many modern concepts are quite within the range of abilities of high school students. Often these notions are easier than traditional material. Here is an opportunity to help our able students prepare for college more effectively.

With these five reasons in mind, I propose to show how some new ideas can revitalize some very old, but still very important, topics in high school mathematics. Let us bring into play the notions of sentences, sets, relations, and functions.

SENTENCES IN ONE VARIABLE

We commonly encounter sentences of two kinds in high school mathematics:

1. Sentences involving names of numbers, or numerals. These sentences are either true or false. For example:

$$1+2=3, \quad 5>3, \quad 6+2=17.$$

2. Sentences involving a variable. These sentences are neither true nor false; they are "true of" or "false of" certain numeral replacements for x. For example:

$$x+3=8, \quad x>5, \quad 2x+1\neq 4.$$

In sentences of the second type, the set of possible replacements for x is the whole set of numbers under consideration in a given context. This set—the totality of numbers under consideration—is sometimes called the *universal set*, and denoted

[2] Interested teachers should write to the Commission on Mathematics, College Entrance Examination Board, 425 West 117th Street, New York 27, New York.

by U. In a given problem, it is important to keep in mind the universal set. For example, suppose that the universal set consists of all positive integers less than 20:

$$U = \{1, 2, 3, \cdots, 20\}.^3$$

Then, in the sentence

$$x+1>12,$$

the possible replacements for x are the 20 integers in U. Of these replacements, the set

$$\{12, 13, 14, \cdots, 20\}$$

makes the sentence

$$x+1>12$$

true; and the set

$$\{1, 2, 3, \cdots, 11\}$$

makes this sentence false. The set of replacements for x that make the sentence true is called the *solution set* of the sentence. Hence, the sentence

$$x+1>12$$

divides the universal set, U, into two subsets: one set containing all replacements for x that make the sentence true (the solution set); the other set containing all replacements for x that make the sentence false.

In high school algebra, the sentences are, for the most part, equations and inequalities. We use the variable as a placeholder for a numeral, which is the name of a number. Now, is not this notion of variable clear and simple? Do we need the additional impedimenta of such expressions as "literal number," "letter-number," "general number," "unknown quantity"? No wonder that many of our students are confused about the meaning of x.[4] I do not suggest that we get into the rarefied upper atmosphere of philosophical discussion on the meaning of "variable": I do say that here is clarification along with simplification.

Thus we have sentences with variables, and these sentences are equations or inequalities. Usually our universal set is the set of real numbers. Here is a fruitful point of view for the high school teacher. Let us think of our sentences (equations or inequalities in x) as selectors of sets: they select from U just that set of numbers that make the sentence true when used as replacements for x. This set of numbers is the solution set of the sentence. Then, if we assume the usual one-to-one correspondence between the real numbers and the points on a line, we can graph the solution set of the sentence. This graph is also called the graph of the sentence. Let us consider some examples:

Example 1. From the foregoing point of view, let us examine the sentence

$$3x+2=6.$$

What is the solution set?

The answer depends on U, the universe of numbers under consideration. If U is the set of positive integers, then the sentence selects nothing: the solution set is empty since there is no integer that can replace x and yield a true sentence.

We have a convenient notation for denoting solution sets, using the set-builder:

$$\{x \mid \ \}.$$

The braces "{ }" are read "set"; the vertical "|" is read "such that." We put the variable on the left-hand side of the vertical bar, and the sentence on the right-hand side. This gives:

$$\{x \mid 3x+2=6\}.$$

Read: "the set of all x such that

$$3x+2=6."$$

If, in Example 1, U is the set of real numbers, then the sentence selects 4/3. Using the set-builder notation, we can denote the solution set thus:

[3] It is common practice to list the elements of a set in braces.

[4] Karl Menger, "Why Johnny hates math," The Mathematics Teacher, XLIX (December, 1956), 578–584.

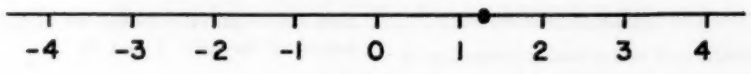


Figure 1

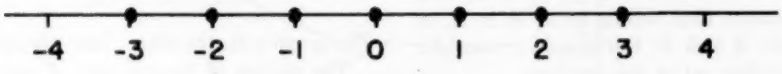


Figure 2

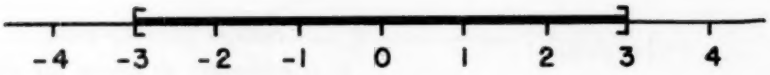


Figure 3

$$\{x \mid 3x+2=6\} = \{4/3\}.$$

The graph of the sentence is made up of just one point. (See Figure. 1.)

Example 2. Suppose that $x^2 \leq 9$ is our sentence. What is the solution set:

$$\{x \mid x^2 \leq 9\}?$$

Again, note the importance of U. If U is the set of integers, then

$$\{x \mid x^2 \leq 9\} = \{-3, -2, -1, 0, 1, 2, 3\};$$

and the graph is Figure 2.

If U is the set of real numbers, then

$$\{x \mid x^2 \leq 9\} = \{x \mid -3 \leq x \leq 3\};$$

and the graph becomes an interval. (See Figure 3.)

Example 3. The sentence

$$x^2-4=(x+2)(x-2)$$

selects everything in the universe:

$$\{x \mid x^2-4=(x+2)(x-2)\} = U.$$

We call such a sentence an *identity*. The graph is the entire number scale, if U is the set of real numbers.

Some observations

The foregoing approach to equations and inequalities (sentences) in one variable places emphasis on their solution sets. In the beginning, these solution sets can be determined by guessing, by experimenting, or by any intuitive method. Soon the need for a more systematic approach becomes evident. This is the time to introduce some general principles and laws for finding solution sets, either by the method of equivalent equations, or by if-and-only-if thinking.[5]

This point of view seems to me to have some definite advantages:

1. It enables us to approach equalities and inequalities together. The neglect of inequalities has been unfortunate in our teaching; some of our best students are handicapped in the calculus because their introduction to inequalities has been too little and too late. Moreover, in some important modern applications of mathematics (linear programming, for example), inequalities are more important than equalities.
2. By identifying the points corresponding to the solution set of a sentence as the graph of the sentence, we broaden and clarify the concept of the graph.
3. Before becoming immersed in the manipulative aspects of solving equations and inequalities, we gain insight into the meaning of the process.
4. We gain some experience in the language of sets.

[5] Robert E. K. Rourke and Myron F. Rosskopf, "What Do We Mean?" The Mathematics Teacher, XLIX (December, 1956), 597–604.

Union and intersection of sets

The notions of union and intersection of sets arise naturally in high school mathematics through this approach. The union of two sets, A and B, is the set composed of elements that belong to A, or to B, or to both A and B. For example, consider the solution set of the sentence

$$x^2>9.$$

If

$$A=\{x \mid x>3\}$$

and

$$B=\{x \mid x<-3\},$$

then the solution set of

$$x^2>9$$

is the *union* of set A and set B:

$$\{x \mid x^2>9\}=\{x \mid x>3 \text{ or } x<-3\}.$$

For the graph of the solution set, see Figure 4.

The *intersection* of two sets, A and B, is the set composed of those elements that belong to both A and B. As an example of the intersection of sets, consider the solution set of the sentence

$$x^2<9.$$

If

$$A=\{x \mid x<3\}$$

and

$$B=\{x \mid x>-3\},$$

then the required solution set is the *intersection* of the sets A and B:

$$\{x \mid x^2<9\}=\{x \mid x<3 \text{ and } x>-3\}$$
$$=\{x \mid -3<x<3\}.$$

The graph tells the story. (See Figure 5.)

The notion of intersection of sets will arise later in connection with the solutions of systems of equations and inequalities.

Sets of ordered pairs

It is customary to denote the coordinates of a point in the Cartesian plane by ordered pairs of numbers such as, for example, (2, 3). The order in which the numbers are written is important. Thus, (2, 3) is not the same ordered pair as (3, 2); their corresponding points are different. In general, we denote by (x, y) a pair of numbers x and y, considered in the order x first and y second. We call x and y the *coordinates* of the ordered pair (x, y).

We now wish to think about sets of ordered pairs. Any set of numbers can yield a set of ordered pairs in the following manner. Let U be a given set of numbers, say

$$U=\{1, 2, 3\}.$$

Then we can form the set of all ordered pairs whose coordinates belong to U. Here's one way of doing it: form all possible ordered pairs with the first coordinate 1; then form all possible ordered pairs with first coordinate 2; and so on. In this operation, the tree graph on the facing

Figure 4

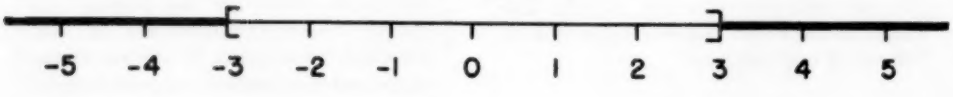
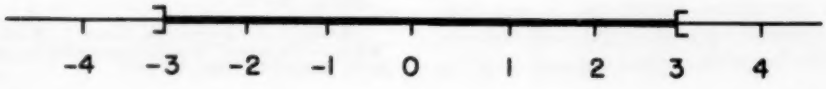


Figure 5



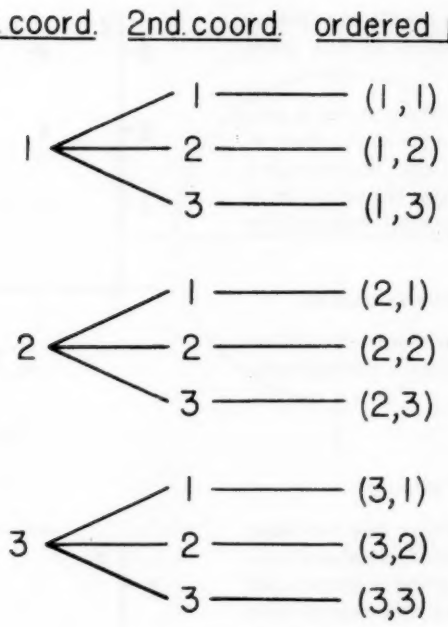


Figure 6

page helps us avoid missing any eligible ordered pair.

The set of ordered pairs obtained in the foregoing manner is denoted by

$$U \times U$$

(read: "U cross U"), and is called the Cartesian set of U. Thus, we have

$$U \times U = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}.$$

We can construct a graph of the set $U \times U$ in the usual way, using the first coordinate of an ordered pair for the abscissa of a point, and the second coordinate for the ordinate. Figure 7 opposite shows the graph of $U \times U$ in the present example. This graph is a square array consisting of 9 points.

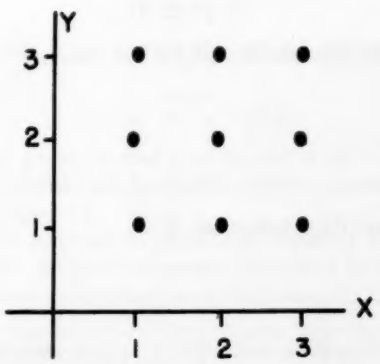
If

$$U = \{1, 2, 3, 4, \cdots\},$$

then the set of ordered pairs $U \times U$ is represented by a lattice of points, including all points in the first quadrant with integral coordinates. Finally, if U is the set of all real numbers, the graph of $U \times U$ is the whole plane. Implicit in the last sentence is the fundamental assumption of analytic geometry:

> *If* U *is the set of real numbers, there is exactly one point of the plane corresponding to each element of the set*

Figure 7



U×U; *and there is exactly one element of* U×U *corresponding to each point of the plane.*

Thus, we may view the plane as a geometric representation of $U \times U$, where U is the set of real numbers. This so-called Cartesian set of U gets its name from Descartes, one of the inventors of analytic geometry.

Sentences in two variables

Sentences in two variables are common in high school mathematics. For example:

$$y = x + 4,$$

$$y > x,$$

$$x^2 + y^2 = 1.$$

Such sentences are neither true nor false. However, they are true of, or false of, certain replacements for x and y. If $x = 1$ and $y = 5$, the sentence $y = x + 4$ is true; if $x = 5$ and $y = 1$, the sentence $y = x + 4$ is false. Such sentences are true of, or false of, *ordered pairs* of numbers. An ordered pair such as (1, 5) that, after replacement (take 1 for x and 5 for y), makes the sentence true, is called a *solution* of the sentence. The set of all such ordered pairs is the *solution set* of the sentence.

Of course, the solution set for a given sentence depends on the totality of ordered pairs under consideration. For example, if the solution set must be found in the Cartesian set $U \times U$, where

$$U = \{1, 2, 3\},$$

then the solution set for the sentence

$$y > x$$

is

$$\{(x, y) \mid y > x\} = \{(1, 2), (1, 3), (2, 3)\}.$$

And the solution set of

$$y = x$$

is

$$\{(x, y) \mid y = x\} = \{(1, 1), (2, 2), (3, 3)\}.$$

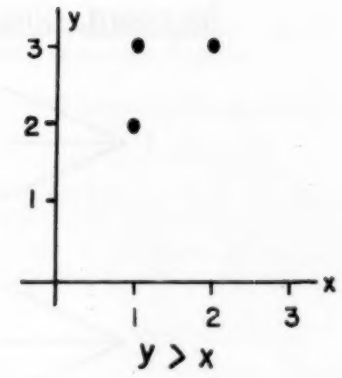


Figure 8

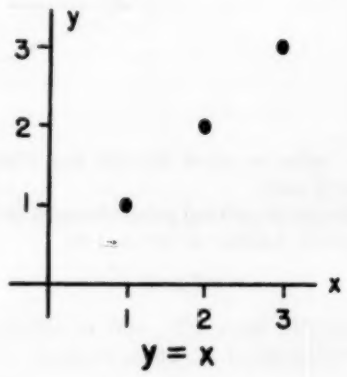


Figure 9

The graphs of these solution sets are shown in Figures 8 and 9.

On the other hand, if U is the set of real numbers, $U \times U$ is represented by the whole plane. The solution sets of $y > x$ and $y = x$ are infinite sets of ordered pairs. The graph of $y = x$ is a straight line, and the graph of $y > x$ is the half-plane above the line that represents the solution set of $y = x$. Figures 10 and 11 show incomplete graphs of these solution sets.

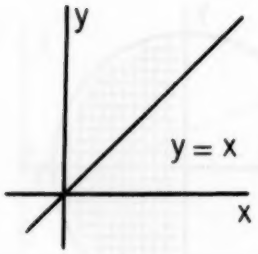


Figure 10

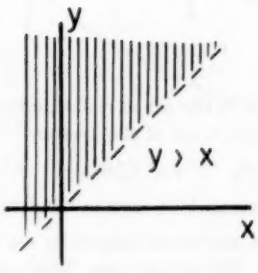


Figure 11

To sum up: if we take U as the set of real numbers, we can regard a sentence in two variables, such as $y=x$, as a *selector of a set*. What is selected is a set of ordered pairs that make the sentence true (the solution set); and this set of ordered pairs is a subset of $U \times U$.

Here is a viewpoint that broadens the idea of a graph, bringing in the graphs of inequalities with those of equalities in a perfectly natural extension. One immediate consequence is a much wider and more interesting variety of graphs, representing vividly the solution sets of equations and inequalities. (If, in addition, we introduce the idea of absolute value, we can further extend the notion of a graph while we throw light on some very important concepts and make solid preparation for the calculus.)

Let us now consider an example from analytic geometry, where U is the set of real numbers.

Example 4. Consider the sentence:

$$x^2+y^2 \leq 16.$$

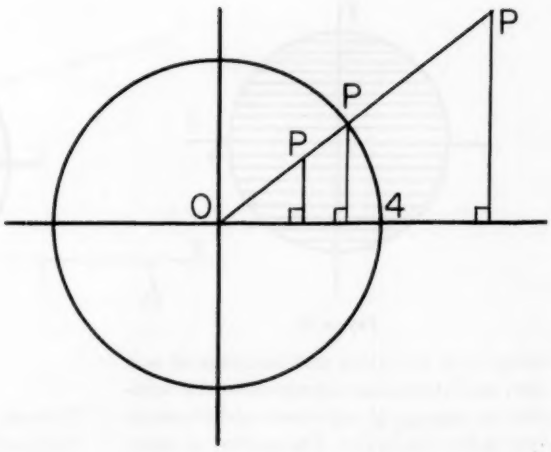


Figure 12

Draw a circle with center at the origin and radius 4 units (Fig. 12). If P is a point with coordinates (x, y), then, since

$$OP^2 = x^2+y^2,$$

we have:

$$x^2+y^2 < 16,$$

if and only if P is inside the circle;

$$x^2+y^2 = 16,$$

if and only if P is on the circle;

$$x^2+y^2 > 16,$$

if and only if P is outside the circle.

The sentence

$$x^2+y^2 \leq 16$$

selects from $U \times U$ the set of ordered pairs:

$$\{(x, y) \quad x^2+y^2 \leq 16\}.$$

The graph of this solution set is the interior and boundary of the circle, as shown in Figure 13.

In addition to providing increased insight, graphs such as the foregoing have recreational value.[6] As a final example, let

[6] Max Beberman and Bruce Meserve, "Graphing in Elementary Algebra," THE MATHEMATICS TEACHER, XLIX (April, 1956), 260–266.

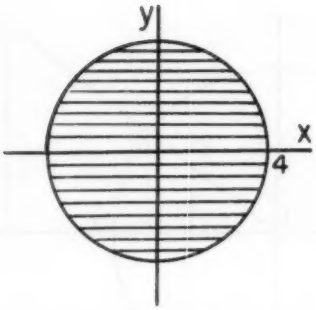


Figure 13

us see how the ideas and language of sets offer an interesting approach to the solution of systems of equations and inequalities in two variables. The notion of intersection of sets is involved.

Figure 14

$A = \{(x, y) \mid x > 0\}$

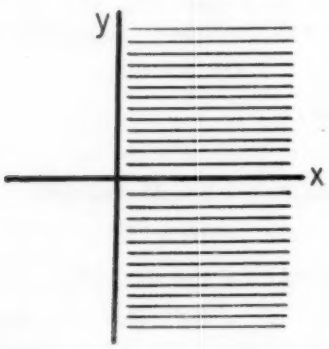
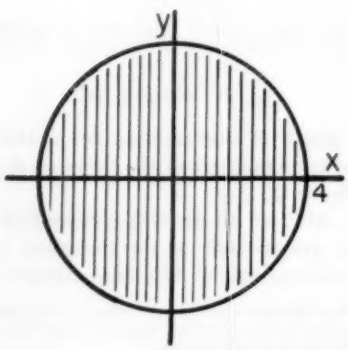


Figure 15

$B = \{(x, y) \mid x^2 + y^2 < 16\}$



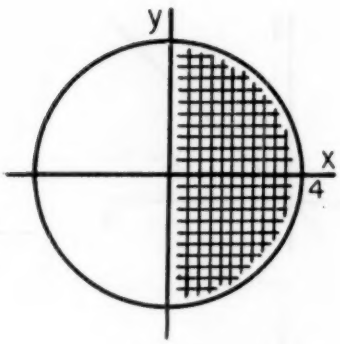


Figure 16

Example 5. If U is the set of real numbers, what is the solution set of the system:

$$x > 0, \quad x^2 + y^2 < 16?$$

(NOTE: The comma here means "and.")

Each of the sentences selects a set of ordered pairs. The graphs are illustrated in Figures 14 and 15.

The solution set of the system is the *intersection* of the two sets A and B. The graph is the interior of a semicircle as shown (Fig. 16).

Relations and functions

In this world of ours, one thing usually depends on another. The idea of a relation is very important in life, just as it is very important in mathematics. C. J. Keyser has written: "To be is to be related, and to know is to see relationships." One of the important contributions of set theory to high school mathematics is the light that it throws on the meaning of a relation, and on special kinds of relations called functions.

Up to this point, we have regarded a sentence in two variables as a selector of a set. For example, the sentence

$$y = x$$

selects from the Cartesian set $U \times U$ (represented by the coordinate plane) a subset of ordered pairs with equal coordinates. The graph is the familiar straight line of Figure 10.

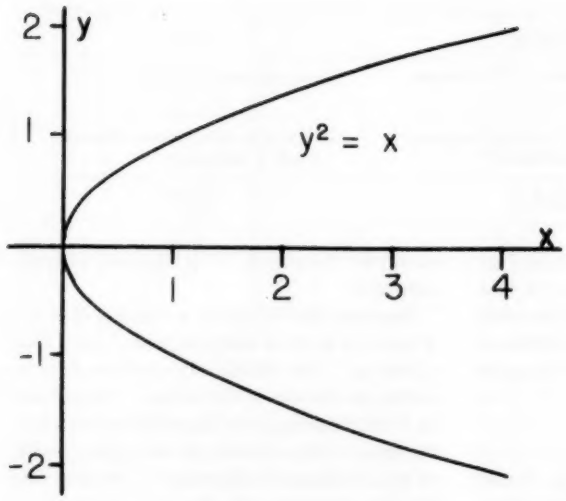


Figure 17

$R_1 = \{(x, y) \mid y^2 = x\}$
(incomplete graph)

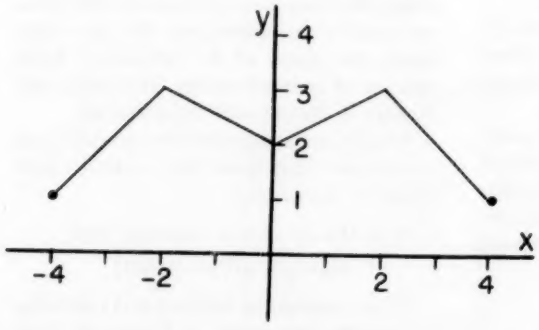


Figure 18

R_2 (defined by a graph)

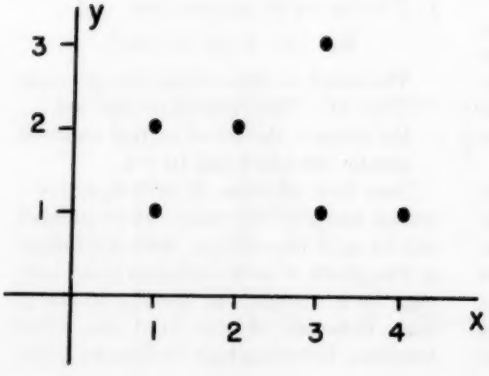


Figure 19

R_3 (defined by a table)

x	1	1	2	3	3	4
y	1	2	2	1	3	1

Relation	Domain	Range
R_1	the set of positive real numbers and zero	the set of real numbers
R_2	the set of real numbers between -4 and 4, inclusive	the set of real numbers between 1 and 3, inclusive
R_3	{1,2,3,4}	{1,2,3}

Now here is another way of looking at things: we may think of a sentence in two variables as *expressing a relation that holds, or fails to hold, for the coordinates* x *and* y *of an ordered pair* (x, y) *belonging to* U×U. Thus,

$$y>x$$

selects from $U\times U$ a subset of just those ordered pairs (x, y) of $U\times U$ for which the relation

$$y>x$$

holds. The graph of this subset is made up of those points of the Cartesian plane whose ordinates exceed their abscissas. (See Figure 11.)

Since the relation expressed by a sentence with two variables leads to a subset of ordered pairs in $U\times U$, let us *identify the relation with the subset.* For example, let us say that the relation in U expressed by the sentence

$$y>x$$

is the subset of $U\times U$ selected by $y>x$:

$$R=\{(x, y) \mid y>x\}.$$

Let us go further and identify all relations in U (whether expressed by sentences or not) with subsets of $U\times U$. Under this agreement, relations are sets—sets of ordered pairs. *By a relation in* U, *we mean a subset of* U×U.

Relations are usually denoted by the letter R. Since relations are sets, we can talk about their unions and intersections; since relations are sets of ordered pairs, we can construct their graphs. A relation may be defined by a formula, by a table of values, or by a graph. Here are some examples in which U is the set of real numbers.

Suppose that we have a relation R in U. Then R is a set of ordered pairs (x, y). The subset in U for which x is a placeholder is called *the domain of the relation;* the subset in U for which y is a placeholder is called *the range of the relation.* As examples, refer to the relations in Figures 17, 18, and 19. For convenience, see the table above.

Note that relations R_1 and R_2 give rise to graphs that are smooth; they have no gaps. The range and domain in such cases are said to be *continuous.* On the other hand, the graph of R_3 contains a finite number of isolated points. The range and domain of R_3 are said to be *discrete.*

Finally, let us consider two special kinds of relations that occur frequently in high school mathematics:

1. U is the set of real numbers, and
$$R_4=\{(x, y) \mid y=2x+3\}.$$
The graph of the relation is the familiar straight line shown in Figure 20. Both the domain and the range are the set of all real numbers.
2. U is the set of real numbers, and
$$R_5=\{(x, y) \mid y=x^2-4x\}.$$
The graph of this relation is a parabola (Fig. 21). The domain is the set U; the range is the set of all real numbers greater than or equal to -4.

Note that relations R_4 and R_5 enjoy a special property: for each x of an ordered pair (x, y) of the relation, there is a unique y. The graph of such a relation is cut only once by a straight line parallel to the y-axis. Relations of this kind are called *functions.* Referring back to Figures 17–19,

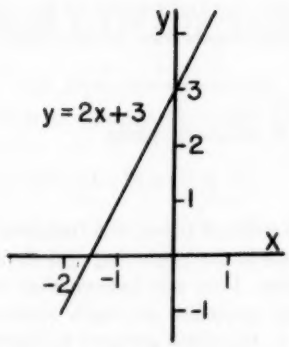


Figure 20

we see that R_2 is a function. Relations R_1 and R_3 are not functions; some of their x's do not have unique y's.

We have arrived at a very important definition: *A function is a set of ordered pairs* (x, y) *such that for each* x *there is exactly one* y. This concept of a function is quite within the comprehension of many high school students, certainly of those who are capable of college. Moreover, this definition of function is easier to teach and gives a much clearer picture than that of Dirichlet. The sentences

$$y=2x+3$$

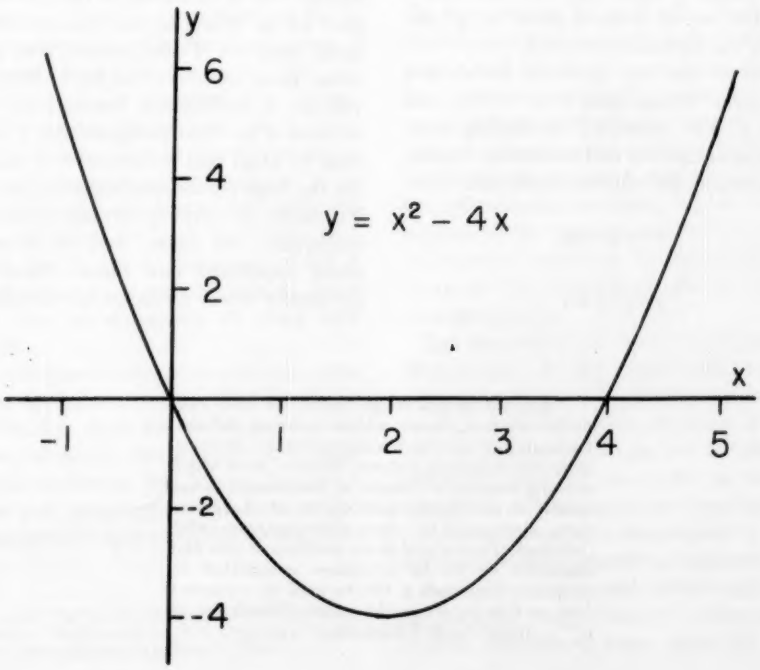
or

$$f(x)=x^2-4x$$

are not functions; they are formulas for functions, or defining equations for functions. Their role is to exhibit a unique y, once the x of the ordered pair (x, y) is known. And the whole subset selected from $U\times U$ by the sentence is the function f.

The notion of function involves a domain, a range, and a rule. The language of sets helps to clarify the meanings of these three words. The domain is the set of all

Figure 21



first coordinates of the ordered pairs that make up the function; the range is the set of all second coordinates of ordered pairs that make up the function; and the rule assigns a unique second coordinate in the range to a given first coordinate in the domain.

Dirichlet (1805–1859) gave the first modern definition of function. In effect, he called the rule the function. The newer definition calls the set of ordered pairs to which the rule gives rise, the function. There is no fuzziness about the meaning of $f(x)$; it is *not* the function, but the second coordinate of the ordered pair whose first coordinate is x.

At the grade ten level, I make the definition of function explicit, and we talk the language of sets and solution sets. We don't get stuffy about it, and we allow ourselves some elliptical expressions. We may, on occasion, call $2x+3$ a linear function, as indeed generations of mathematicians have done and will likely continue to do for some time. But we keep in mind that this is elliptical talk, that the function is really the set of ordered pairs (x, y) defined by the formula $y=2x+3$.

I believe that my students are clearer about some things than ever before, and so am I. For example, in dealing with quadratic equations and quadratic expressions, certain differences stand out. Consider

$$x^2-4x=0$$

and

$$y=x^2-4x.$$

When one thinks in terms of the solution sets of these sentences, he sees that

$$\{x \mid x^2-4x=0\} = \{0, 4\},$$

a pair of numbers; while

$$\{(x, y) \mid y=x^2-4x\} = f,$$

a set of ordered pairs, or a function.

I spoke at the beginning of difficulties in semantics. I do not believe that anyone seriously proposes to teach abstract set theory in the ninth grade or in the twelfth grade. Nor is symbolic logic, in full college array, suitable for grade ten. I have been discussing topics that are important to traditional mathematics and to new mathematics in high school: equations, inequalities, graphs, variables, relations, and functions. In each of these topics, the concepts and language of sentences, sets, and relations have something worthwhile to offer.

It is not the job of the high school teacher to teach the subject matter of the college, but it definitely is our job to prepare for it. When we can look ahead and make good use of what we see, then we become more effective teachers. When the voltage is sufficiently lowered on many notions of modern mathematics, I believe that we shall find wonderful new material for the high school mathematics program. To clarify, to simplify, to unify, to broaden important old ideas, and to introduce some important new ideas—these goals are surely worth a change in viewpoint.

"Instead of this single unity, we offer children—Algebra, from which nothing follows; Geometry, from which nothing follows; Science, from which nothing follows; History, from which nothing follows; a Couple of Languages, never mastered; and lastly, most dreary of all, Literature, represented by plays of Shakespeare, with philological notes and short analyses of plot and character to be in substance committed to memory. Can such a list be said to represent Life, as it is known in the midst of the living of it?"—*Alfred North Whitehead.*

Sweet are the uses of generality[1]

ROBERT A. ROSENBAUM, *Wesleyan University, Middletown, Connecticut.*

Abstraction and generalization lie at the heart of mathematics. Let us not debase our subject by ignoring these aspects. . . .

IN MY TITLE, *generality* isn't precisely the right word with which to replace Shakespeare's *adversity;* indeed, I don't think that any single word expresses what I have in mind, but *generality* is the important aspect of *abstraction* for the thesis that I wish to develop.

Let me begin by reconstructing two polar views of mathematics. The first is exemplified by Lancelot Hogben, who says something like this: "All significant mathematics has developed *directly* out of the necessity of solving a practical problem. Mathematics created in the ivory tower is bound to be sterile, unproductive, and unimportant." The opposing view is contained in the traditional toast of the pure mathematician, "Here's to mathematics; may she never be of use to anybody." I've never heard any mathematician offer this toast, attributed to H. J. S. Smith; perhaps G. H. Hardy, in the spirit of his *A Mathematician's Apology,* might have come close to doing so, at least half seriously.

I would like to suggest a position intermediate to these extremes. Historically, it is undeniable that the development has been stimulated by the necessity of solving practical problems. But human beings, a curious and imaginative breed, have not been satisfied to stop when the solution has been found. Ideas which have grown out of the original problem are pursued "for their own sake." Ramifications of these ideas may lead so far that it may be difficult to reconstruct the role of the original problem in setting off the chain of mathematical development. Whether the practical origins are considered or not, the mathematics has become "pure"—highly abstract, a study of relations, devoid of content.

But it probably doesn't remain "pure." To the dismay of a G. H. Hardy, the mathematician's private game becomes a public tool. We have many familiar examples of this situation. The geometry of the conic sections, as developed by the Greeks, proved to be of considerable use to Newton in describing planetary motion. The algebra of complex numbers, which had its origins in the (pure) problem of solving polynomial equations, furnished the technique for the analysis of alternating current machinery.

Let me add a less hackneyed example. Weierstrass, in the nineteenth century, gave an example of a continuous function which has a derivative nowhere: a curve with no breaks or jumps, but so infinitely "crinkly" that at no point is there a tangent line. The invention of such a function was greeted with amazement by many mathematicians—intuition indicates that a continuous curve may have many cusps or corners, but "clearly" always has smooth sections at every point of which

[1] A paper read at the meeting of the Association of Teachers of Mathematics of New England at North Andover, Massachusetts, on May 3, 1957.

there is a tangent. No tangents at all! Surely the mathematicians are going too far when they spend time on such esoteric, such pathological examples!

I confess that I have sometimes apologized to students when mentioning this example—"Of course, fellows, you'd never run into anything like this in practice, but take a look at what crazy mathematicians have cooked up." Now I read in the second volume of Norbert Wiener's autobiography that the nowhere-differentiable function proved necessary to his analysis of Brownian motion.

I'm suggesting that problem and solution, practice and theory, reality and abstraction aren't *separate* concepts in the history of mathematical development—they go together like love and marriage, in the recent popular song. Indeed, as love leads to marriage, and marriage extends the dimensions of love between husband and wife, so likewise does a practical problem lead to a mathematical solution, whose ramifications and extensions may lead to deeper insights into the original problem, and to the solution of new problems. These, in turn, suggest fresh avenues for exploration and the development of new mathematics, and so on.

Thus, these two aspects of mathematics constitute members of a "mutual benefit society," each helping the other. The engineer considers mathematics to be ancillary to his discipline; the mathematician looks at the situation in reverse fashion. The relative importance of the two aspects is not significant—the fact is that each is useful to the other.

Indeed, many of our greatest mathematicians consider that a relationship between their work and applications is not only useful, it is actually essential, if their work is to be fruitful. In holding this view they are not necessarily agreeing with Hogben, for they do not insist that their mathematics grow directly out of a practical need; but they feel that their work, to be of prime significance, must have some points of contact with reality.

Surely the mathematicians of recent years who are generally acclaimed as outstanding—Poincaré, Weyl, Wiener, Einstein, von Neumann, to name just a few—have all heeded the discipline of reality, no matter how *abstract* their work might be. Wiener testifies that considerations of practical problems have always been in his mind and have stimulated the *best* of his theoretical work. Von Neumann goes so far as to state that mathematics too long divorced from reality tends to become baroque—grotesque, corrupt, and decadent.

The converse proposition—that mathematics is useful and indeed essential to the solution of practical problems of science and engineering—will receive ready, universal assent. *In my opinion the assent is often given for the wrong reasons, which causes us to give improper emphasis in our teaching, and to foster unreasonable and unfortunate attitudes toward mathematics in our students.* This is the thesis which I would like to elaborate and to defend.

A teacher of high school physics who spends his class time repairing vacuum cleaners and such like finds that his students enjoy the classes and that their parents are delighted to have them around the house—unlike the situation with many teen-agers. But these students aren't learning physics. Unforunately neither they nor their parents realize this fact.

We can find comparable circumstances in the mathematical field. A student who is a bear-cat at factoring but who doesn't understand the distributive axiom doesn't know algebra—and he isn't even much help around the house: after all, most households do manage to get along without frequent factoring.

I don't mean to say that the repair of vacuum cleaners has no place in a physics course. In the first place many people have a great enthusiasm for gadgeteering, and the motivational advantages of introducing physics through vacuum cleaners should not be underestimated. Moreover, a student who can't apply his knowledge of

physical principles to the repair of vacuum cleaners hasn't learned all there is to know. But if the repair of vacuum cleaners is carried out without reference to basic principles of physics, the student isn't learning *physics*, he's learning a *trade*—and he isn't even learning a trade very well.

Likewise, in mathematics, the drill of factoring and other techniques is essential in the learning of the subject. But this drill must not replace the understanding of the *reasons* for the operations. Indeed, an understanding of reasons helps to cut down on the amount of routine drill needed to give the student technical competence. Time after time I've heard students say, "Now that I understand what I'm doing when I cross-multiply [or add two fractions, or differentiate under the integral sign], I find that I can work the problems quickly without trouble." We tend to underestimate the desire of students to know "why" and their capacities to absorb explanations.

As students approach maturity they seem to lose some of their youthful curiosity. Perhaps this is a natural human tendency, and perhaps it's fostered by the atmosphere of our culture, including, I fear, the attitudes expressed by us teachers.

Whatever the reasons, many of my college students seem to be thinking, "I'm going to be an engineer [or a nuclear physicist]. I'm going to need mathematics in my profession—I'll have to find moments of inertia, and centers of pressure, and total charge on a condensor, and families of solutions of differential equations subject to boundary conditions. I need good, practical calculus to help me solve these problems. My teacher's a nice guy, and his vaporings on each branch of mathematics as a hypothetico-deductive system are rather interesting, but I wish that he'd understand that I can't afford to waste much time on idle chatter. That sort of stuff is O.K. for a mathematician, but I'm going to be a scientist."

What happens when these students become professional engineers? For one thing, they probably echo the familiar remark, "I've never used 'calc' since I left college." (This may be said wonderingly or defiantly.) But—and this is the distressing thing—whenever they encounter a novel problem, for which a handbook solution is not readily available, they have to turn to a mathematician for consultation.

There was a time when an engineer or scientist who knew how to perform the rudiments of differentiation and integration could be a leader in his field. Some of us have seized on this idea and won't let it go. Let's teach our students the basic techniques of the calculus, we say; let's give them lots of drill in finding centers of gravity, moments of inertia, and torques, and we'll have done our duty as mathematics teachers. In this respect we're like the fellow who played the same note on his cello, hour after hour, day after day. In explanation of his idiosyncrasy he remarked, "Some people fiddle up and down, fingering madly, looking for a note that pleases them. I've found it."

Unfortunately for us, "all is flux." Neither mathematics nor its application is static. (If my introductory argument about mutual interaction has validity, we should *expect* continual change.) Tables, handbooks, and high-speed computing machines have solved the problems to which the basic techniques of the calculus are adapted. If we teach the basic techniques of the calculus, *and nothing more*, we are following the example of the oolii-ooli bird who always flies backward because he doesn't give a hoot where he's going, but likes to see where he's been.

What else, then, should we teach? One possible choice is matrix theory. Matrix theory, as one part of linear algebra, is an important branch of mathematics which happens to appeal to me greatly. Therefore, I enjoy sharing my enthusiasm for the subject with my students. More than this, matrix theory has numerous applications. One of the standard books on the

subject was written by three British aeronautical engineers—as a text for other aeronautical engineers. Psychologists, sociologists, and economists—at least those with certain predispositions—use matrix theory as a basic tool. When I teach the subject I feel the glow that comes from being a "man of the world." This is stuff that's really *used*, not just by the academic researchers in other recondite mathematics, but by the up-to-the-minute designers of airfoils and nozzles.

Once again I find myself in a fool's paradise. Matrix theory does not represent the be-all and end-all of mathematics. To teach matrix theory, *and nothing more*, is a trade-school occupation. I'm teaching the trade of a machine which isn't yet obsolete, but it's obsolescent, or on the way to becoming so. Lest my remarks be misunderstood, I emphasize that calculus isn't obsolete. Nor do I object to the inclusion of geometry *and* algebra *and* calculus *and* matrix theory in the curriculum. What I'm objecting to is the *spirit* in which they are often presented.

What spirit do I advocate? What is needed, I think, is an appreciation of the way in which mathematicians attack new problems. The freedom and imagination that the real mathematician brings to bear on a hard problem are the source of his success, both in mathematics and in applications thereof. This attitude explains Wiener's triumphs in areas where well-trained engineers and physicists had failed.

The contrary attitudes of orthodoxy and rigidity inhibit mathematical development. Let me give two examples.

(1) Consider the equation $x+(1/x)=1$. This clearly has no solutions. For x must obviously be positive if we are to have a prayer of finding a root. However, the sum of any positive number and its reciprocal is greater than one. Thus, no solutions.

What's the difficulty with this argument? Nothing, if I'm assuming that I'm going to consider "numbers" which are linearly ordered. But if I'm willing to *extend* the number system by inventing complex numbers, I'm in a different world. As I remarked earlier, engineers are grateful for this invention.

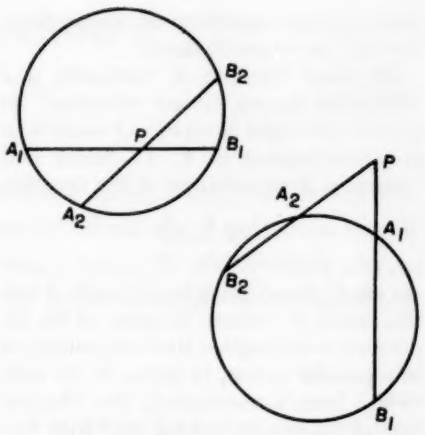
(2) According to legend, Hamilton was stymied for a time in the process of inventing quaternions. Most of his development was satisfactory, but some of it was incompatible with an "obvious" relation, $p\times q=q\times p$. Next to our traditional death, taxes, and $2+2=4$, nothing is surer to the layman (if he thinks about it at all) than the commutative property of multiplication. Mathematicians, too, were inclined to assume that multiplication is necessarily commutative. But Hamilton solved his problems in radical fashion—he defined his multiplication so that it *wasn't* commutative. Matrix multiplication isn't commutative, either, and even aeronautical engineers now agree, cheerfully or reluctantly, that $p\times q$ is not necessarily equal to $q\times p$.

The mathematician's freedom to invent gives scope to his imagination and eventually accounts for his success. There is an old baseball story which illustrates my point. Three umpires were discussing their own attitudes and virtues. The first umpire remarked, "Some is balls and some is strikes, and I calls 'em as I sees 'em." The second, a bit more self-assured, said, "Some is balls and some is strikes, and I calls 'em as they are." Then up spoke Bill Klem, the dean of them all: "Some is balls and some is strikes, but 'til I calls 'em, they ain't *nothing*." The first of these umpires, wrestling with reality to the extent of his own poor ability, is clearly the engineer type. The second, describing the universe in sure, bold strokes, is a close relative of a happy physicist—of the nineteenth century. But Klem takes his place with *other* mathematical immortals—Bolyai, Lobachevsky, Riemann—who have invented their own worlds.

The willingness to consider systems satisfying arbitrary assumptions gives the mathematician power to abstract from individual cases to formulate a general theory encompassing the special cases. For every situation he must be able to see analogies and generalizations, similarities and specializations—the more the better. If he does not agree with Gilbert that "things are seldom what they seem" he must at least be prepared to admit that things *need* not be what they seem. He must look for unity in diversity—his motto might be E Pluribus Unum. In short, he must be imaginative.

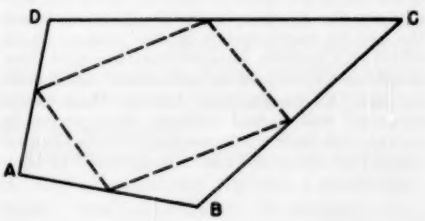
All too frequently in my teaching, as I indicated in some of my earlier remarks, I neglect the generalization for the specific fact, the theory for the application. Other teachers have this failing too—indeed, my favorite horror story involves just this point. Some years ago I participated in a teachers' "workshop" in which the teachers were encouraged to present interesting enrichment material with which they were familiar. One high school teacher remarked that, to square $3\frac{1}{2}$, one could multiply 3×4 and add $\frac{1}{4}$; to square $4\frac{1}{2}$, one could multiply 4×5 and add $\frac{1}{4}$. I confessed that I wasn't acquainted with this short-cut and offered to check whether it was true in general: I wrote on the board: $(n+\frac{1}{2})^2 = n^2+n+\frac{1}{4} = n(n+1)+\frac{1}{4}$. "Well," I said, "this is a general formula then—to square any integer plus $\frac{1}{2}$ we multiply the integer by the next larger and add $\frac{1}{4}$." "Yeah," responded my friend who had made the initial suggestion, "and it works for $5\frac{1}{2}$, too."

Surely the illustration I've just cited must be unique —this must be the nadir of American mathematical education. But there are innumerable examples in elementary mathematics where we fail to take advantage of opportunities to generalize, to show the relation between superficially distinct problems, to encompass several cases in a single description. Let me give three rather common examples of such instances.



There is the familiar theorem that the product of segments of chords through a point within a circle is constant. Another standard theorem of plane geometry states that, for a given point external to a circle, the product of each secant by its external segment is constant. From any half-way sophisticated point of view, these are not distinct results: properly, for any position of P, $PA_1\times PB_1 = PA_2\times PB_2$. From the point of view *of applications*, the principal reason for deriving theorems about circles is to obtain results which are at once true of wheels, and of round tables, and of coins, and of manhole covers. We would not think of stating a separate result for each of these objects. Why are we satisfied to present two theorems about circles when one will do?

Another example relates to a geometric theorem true both in two and in three dimensions. If we join in order by straight lines the midpoints of the sides of the quadrilateral $ABCD$, we obtain a parallelogram. This is true whether $ABCD$ is a



plane or a skew quadrilateral. Do we point this fact out to our students?

My third example of "analogies and similarities among diverse situations" is not one I thought of myself—I found it in an advertisement of D. C. Heath and Company. Four instances of the equation

$$\frac{1}{x}=\frac{1}{x_1}+\frac{1}{x_2}$$

are cited: the effective focal length of two thin lenses in contact, in terms of the individual focal lengths; the total resistance in a parallel circuit, in terms of the individual branch resistances; the effective rate of filling a swimming pool with two inlets; and the effective speed of an aircraft in flying away from its base and returning, if there is a wind blowing.

Abstraction and generalization lie at the heart of mathematics. Let us not debase our subject by ignoring these aspects in the misguided notion that, by so doing, we can increase the practical value of mathematics.

The quotation from Shakespeare goes like this:

> "Sweet are the uses of adversity
> Which, like the toad, ugly and venomous,
> Wears yet a precious jewel in his head."

I would hope that we can convince our practical-minded students that mathematical generality, far from being ugly and venomous, is actually beautiful and beneficent. Whether we succeed in this effort or not, we surely should be able to convince them that generality is the jewel in the crown of mathematics.

What we must realize above all is that the mathematical education of the past has to a disturbing extent failed to lay the groundwork for the kind of intercommunication among mathematics, the various sciences, and engineering, which we now see to be necessary. By and large mathematical instruction has been little touched except at the graduate level, by the mathematical advances of this century. Until it is, such improvements as are made in it will be mainly of a technical or pedagogical nature. The most serious obstacle to a modernization of the mathematical curriculum is the utilitarian spirit which pervades secondary education and governs the manner in which scientists teach the use of mathematics in the various fields where it is applied. Because students have been taught in high school to understand mathematics more in its practical aspects than in its technical and logical fullness, they arrive in college with their mathematical abilities blunted instead of sharpened and strengthened as they should have been. They are further encouraged to take a utilitarian view of mathematics by the way they see it handled—and, at times, mishandled—in nearly every scientific or engineering course they may elect. The consequence is that the attempt to teach calculus properly or to introduce the elements of modern algebra into the curriculum is often resented by students and criticized by other departments. Despite this difficulty some progress has been made. Much more, however, is essential before American mathematics can be considered to be in a sound and healthy state. For the words of Francis Bacon surely have their application to this matter—"If you will have a tree bear more fruit than it used to do, it is not anything you can do to the boughs, but it is the stirring of the earth and the putting of new mould about the roots that must work it."—*Taken from "Mathematics and the Future of Science," by Marshall H. Stone,* Bulletin of the American Mathematical Society, *Volume 63, Number 2, March 1957.*

Finding out about "modern mathematics"

KENNETH O. MAY, *Carleton College, Northfield, Minnesota.*

What is this "modern mathematics" about which we hear so much? In this article some easily accessible and readable sources of information are briefly described.

IN THE LAST few years there has been considerable discussion about the possibility of introducing into the secondary school curriculum topics such as logic, theory of sets, Boolean algebra, and the set theoretic approaches to relations, functions, and other topics of elementary mathematics. For lack of a better term such material has come to be called "modern mathematics." It is only in very recent years that these topics have appeared in college courses; and, until very recently, there were virtually no books dealing with this material at the elementary level. Hence it is inevitable that few teachers are familiar with it. However, several books written recently for use in elementary college courses could also be used by high school teachers as sources of information on the ideas involved and even as sources of materials that could be used in the high school classroom.

Principles of Mathematics by C. B. Allendoerfer and C. O. Oakley (New York: McGraw-Hill Book Co., Inc., 1955) is in use as a freshman text in a number of colleges and could be used in high school classes. The first six chapters treat logic, the number system from a modern point of view, groups, fields, sets, Boolean algebra, relations, and functions. The treatment is not profound and will not answer all the teacher's questions, but it is a very satisfactory introduction to the ideas of modern mathematics.

A second book at about the same level of difficulty is *Introduction to Finite Mathematics* by J. G. Kemeny, J. L. Snell, and G. L. Thompson (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957). This book makes no attempt to discuss traditional topics and so is able to cover a wider range of modern material. It deals with logic, sets, applications of set theory to probability, vectors, matrices, linear programming, theory of games, and applications of these ideas to social science problems. Like the previous book, it could it could be used in high school.

Concepts and Structures of Mathematics (A Syllabus) by the Mathematics Staff of the College of the University of Chicago (Chicago: University of Chicago Press, 1954) is a more thorough and profound treatment of the basic ideas of sets and logic together with their application to the notions of function, relation, and the construction of deductive theories. The Mathematics Staff at the College did the pioneering work in this area and showed that material of this kind could be taught to young people with very little mathematical background. This book represents a revision of a textbook that they have used with students of high school age over a period of years at the College. It deserves very careful study.

Fundamental Mathematics by Kenneth O. May (Preliminary Edition, Cambridge, Mass.: Addison-Wesley Publishing Co.,

Inc., 1955) is designed as an elementary college text from the modern point of view. Its level of difficulty is about the same as the University of Chicago text and somewhat higher than the previous two books mentioned above. It devotes more space than the books previously mentioned to careful discussion of the meaning of terms, the distinction between symbols and the things they stand for, and the nature of proof. In its present form it is not suitable as a high school textbook, but high school teachers might find it useful for study and reference.

Edmund C. Berkeley has written two brief pamphlets that the high school teacher may find quite helpful. They are *Circuit Algebra, Introduction* (New York: Edmund C. Berkeley and Associates, 1952) and *A Summary of Symbolic Logic and Its Practical Applications* (New York: Edmund C. Berkeley and Associates, 1954). These treatments are too brief to be used alone, but they are suitable for supplementary reading by teachers or students.

Another book that is very valuable for study and reference is *Universal Mathematics, Part II, Structures in Sets*, A Joint Project of: The Committee on the Undergraduate Program of the M.A.A.; the 1954 Kansas Summer Group; and a Tulane University Writing Group (New Orleans, Louisiana: Tulane Book Store, Tulane University, 1955). This book gives a much more detailed treatment of sets and applications of set concepts than any of those mentioned above. Part I of this work presents the concepts of functions and limits from a novel and very modern point of view, but is not likely to be helpful to the high school teacher looking for material for use in class.

There are other elementary college textbooks that use these so-called modern ideas to a greater or lesser degree, but we are not attempting to be complete here. For the teacher who becomes familiar with the content of the books mentioned above (and for this purpose it is not necessary to read them all) the next step might be to study books that deal more thoroughly with logic and the foundations of mathematics. An excellent book of this kind is *Introduction to Logic and to the Methodology of Deductive Sciences* by Alfred Tarski (New York: Oxford University Press, 1946). It is not very difficult reading and gives a thoroughly sound treatment of logic from a modern point of view. There are many other books on mathematical logic that would be readable for anyone who was familiar with Tarski's classic treatment.

An elementary introduction to the structure of mathematics is *An Introduction to Mathematical Thought* by Edward R. Stabler (Cambridge, Mass.: Addison-Wesley Publishing Co., 1953). This book deals with mathematical thought in relation to logic and science, including the nature of mathematical knowledge, the origin and influence of logical systems, the essentials of logic, and scientific method. It also gives a number of examples of deductive mathematical systems. Much of the discussion is relevant to material treated in high school mathematics classes, and the book is not difficult to read.

It would be possible to mention a number of books suitable for the teacher who is looking for something harder to chew upon. An excellent book of this kind is *Introduction to the Foundations of Mathematics* by Raymond L. Wilder (New York: John Wiley and Sons, Inc., 1952). Here one will find a more careful and complete treatment of the axiomatic method, the theory of sets, infinite sets, transfinite numbers, and the different philosophical points of view on the nature of mathematics.

The books listed above have been designed for college students at various levels. As indicated, some of them are suitable for reading by high school students. However, as far as this writer knows, the only materials now available specifically designed for the high school student and written from the modern

point of view are those being prepared by the University of Illinois Committee on School Mathematics. Teachers who can secure the teacher's edition of *High School Mathematics—First Course* (ninth grade) from the Mathematics Project at the University High School, Urbana, Illinois, will see one way in which the ideas of modern mathematics can be adapted to high school use. And this writer can testify from personal observation that these materials do work in high school.

"Our brightest teen-agers are not bookworms," according to a study of the National Merit Scholarship Corporation. "They are able students with a surprising number and variety of extracurricular and community activities. They come from every rank and station in the economy." The average award, based on need, is $630 per year for four years.

Of the 556 winners, only 71 came from small high schools (graduating class of 50 or less), and 82 came from schools which have an annual group of 500 or more seniors. The majority of their career choices call for technical preparation: 29% expect to be researchers, 23% engineers, and 17% teachers.—NEA Journal, *March 1957*.

We have the strongest logical and historical grounds for believing that the tendency towards the mathematization of knowledge, which began with the Greeks, will be broadened and accelerated in the coming century. That this tendency will require and stimulate advances in mathematics itself is inevitable. To some extent it will even be directed and conditioned by the progress which can be made by mathematicians, as we can infer from a consideration of the history of modern physics. The future of science is in this sense tightly bound together with the future of mathematics. The extraordinarily luxuriant growth of pure mathematics in our time encourages us to look with optimism and enthusiasm to a future in which science will draw bountiful nourishment from the fruits of our husbandry in the mathematical field.—*Taken from "Mathematics and the Future of Science," by Marshall H. Stone,* Bulletin of the American Mathematical Society, *Volume 63, Number 2, March 1957*.

Four more exercises in cutting figures

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This article provides still other new exercises for the student of plane geometry, and in so doing adds to the repertory of methods developed in five earlier papers on the same theme.

IN THE MAY 1957 issue of THE MATHEMATICS TEACHER we published the last in a series of five papers[1] on the cutting of figures. We were content to put the subject aside at that point because we felt our little series rounded it out rather handily. Of course, much of value can be added. The four small problems solved here are, we feel, interesting in themselves and suggestive as supplements to our series. We call these problems "Exercise 12," "Exercise 13," "Exercise 14," and "Exercise 15," intending thereby to indicate both the nature of the present material and its relation to the May paper and to the series generally.

EXERCISE 12 AND ITS SOLUTION

Exercise 7—the last exercise discussed in our February paper—dealt with the transformation of an arbitrary right triangle into a square. The general method for effecting this transformation is: *first*, transform the triangle into a rectangle (by one cut, along the segment joining the midpoint of the hypotenuse with the midpoint of one of the sides); and *second*, transform the resulting rectangle into a square (e.g., by methods developed in Exercise 6 of our February paper).

For certain types of right triangle, however, this general method is not the most economical, that is, it does not give the smallest numbers of parts. For example, a right triangle ACB whose longer side BC is less than twice its shorter side AC (see Figure 1) can be transformed into a square

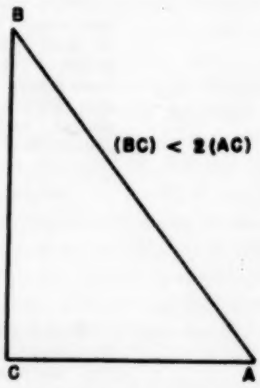


Figure 1

without cutting ACB into more than four parts. Some importance attaches to special methods of this sort because pieces of material (like unworked leather) often are similar to triangle ACB, and their *economical* transformation into squares reduces the costs of mass production (of, say, shoes).

Exercise 12: Given a right triangle ACB *whose longer side* BC *is less than twice its shorter side* AC, *to cut it into no more than four parts that reassemble into a square.*

Solution of Exercise 12: Let us begin with the right triangle ACB of Figure 1, assuming only that

[1] "A Problem on the Cutting of Squares," "More on the Cutting of Squares," and "Still More on the Cutting of Squares," THE MATHEMATICS TEACHER, XLIX (May, October, December, 1956), 332–43, 442–54, 585–96; and "New Exercises in Plane Geometry" and "More New Exercises in Plane Geometry," THE MATHEMATICS TEACHER, L (February, May, 1957), 125–135, 330–339.

$$(1) \qquad 1 < \frac{(BC)}{(AC)} < 2.$$

(Statement (1) sees the use of our convention that (XY) denotes the length of segment XY.) The area of triangle ACB is $\frac{1}{2}(AC)\cdot(BC)$, hence the square sought must have a side of length

$$(2) \qquad \sqrt{\frac{(AC)\cdot(BC)}{2}}.$$

In triangle ACB, draw the segment ED joining the midpoint E of the longer side BC with the midpoint D of the hypotenuse. Noting that

$$(3) \qquad (EC) = \frac{(BC)}{2},$$

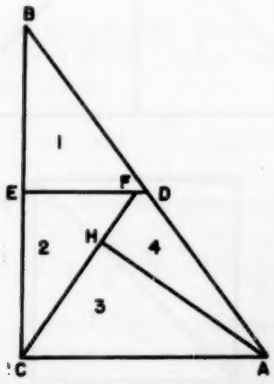
let us construct a segment of length (2) by constructing the mean proportional between (AC) and (EC). With this segment as radius and C as center, mark the point F on ED such that

$$(4) \qquad (FC) = \sqrt{\frac{(AC)\cdot(BC)}{2}}.$$

Draw segment FC. Finally, draw segment AH perpendicular to FC. As Figure 2 shows, the three segments ED, FC, and AH divide our original right triangle ABC into four parts. These four parts reassemble into a square.

Before showing that this construction solves Exercise 12, we must verify that the construction is actually possible; i.e., we must show that there is precisely one point F on ED such that (FC) equals the length of a side of the square equivalent to triangle ACB. To this end we shall establish that (i) $(EC)<(FC)$ and (ii) $(EF)<(ED)$.

Figure 2



Consider (i). From (1) we know that

$$\frac{(BC)}{2} < (AC),$$

hence by (3) that $(EC)<(AC)$. Now (FC) is the mean proportional between (EC) and (AC). Thus $(EC)<(FC)<(AC)$, and in particular $(EC)<(FC)$ as desired.

Consider (ii). From (4) we have that $(FC)^2=\frac{1}{2}(AC)\cdot(BC)$. Since triangle CEF in Figure 2 is a right triangle, it follows that

$$(5) \qquad \begin{aligned}(EF)^2 &= (FC)^2-(EC)^2 \\ &= \frac{(AC)\cdot(BC)}{2} - \frac{(BC)^2}{4}.\end{aligned}$$

Appealing again to (1), we see that $(AC)<(BC)$ and so $(BC)-(AC)>0$. Therefore $((BC)-(AC))^2>0$, which is to say

$$(BC)^2+(AC)^2-2(AC)\cdot(BC)>0;$$

and by transposing terms and dividing through by 4:

$$\frac{(AC)^2}{4} > \frac{(AC)\cdot(BC)}{2} - \frac{(BC)^2}{4}.$$

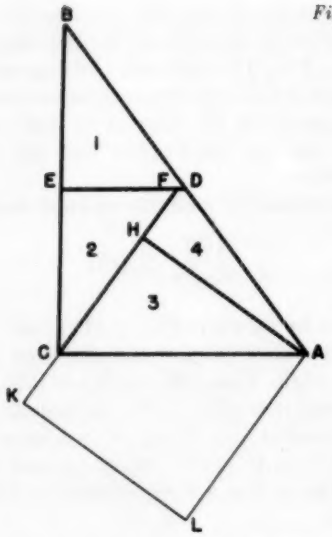
This last, in view of (5), tells us that

$$\frac{(AC)^2}{4} > (EF)^2,$$

and consequently that $(EF)<(AC)/2$. Since $(ED)=(AC)/2$, it follows that $(EF)<(ED)$ as desired.

Let us return to the matter of showing that the four parts 1, 2, 3, 4 of Figure 2 actually reassemble into a square. First, add to Figure 2 and obtain Figure 3 by constructing the square $AHKL$ having segment AH as one side. This square $AHKL$ has an area equal to the area of our original right triangle ACB. For in view of

Figure 3



the similarity of triangles AHC and ECF,

$$\frac{(AH)}{(AC)}=\frac{(EC)}{(FC)};$$

consequently, by (3) and (4),

$$(AH)=\frac{(AC)(EC)}{(FC)}$$

$$=\frac{(AC)\cdot(BC)}{2}\Big/\sqrt{\frac{(AC)\cdot(BC)}{2}},$$

and so

$$(AH)=\sqrt{\frac{(AC)\cdot(BC)}{2}}.$$

It is an easy matter, finally, to show that parts 1, 2, 3, 4 of triangle ACB can reassemble into the square $AHKL$ of Figure 3. First, add to Figure 3 and obtain Figure 4 as follows: draw segment LM perpendicular to AC, segment LN parallel to AB, and label the resulting four parts of square $AHKL$ as shown. Now part 3 is evidently common to the square and the triangle. Further, it is not difficult to prove that each pair of parts labeled with the same numeral is a pair of congruent parts. In this fashion we establish that the four parts of triangle ACB reassemble into a square. Exercise 12 has been solved.

Remark on the solution of Exercise 12. While Exercise 12 concerned right triangles ACB subject to (1), our solution of it also indicates a procedure in case $(AC)=(BC)$, i.e., in case triangle ACB of Figure 1 is an isosceles right triangle.

Figure 4

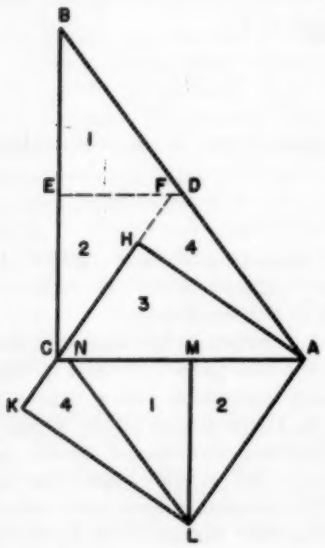
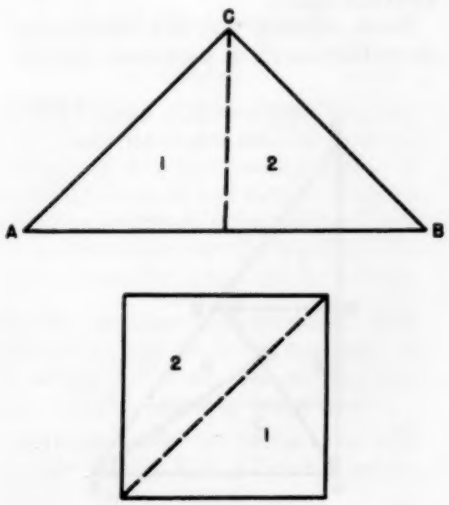


Figure 5



When $(AC)=(BC)$, it follows from (5) that

$$(EF)^2=\frac{(AC)^2}{2}-\frac{(AC)^2}{4}=\frac{(AC)^2}{4}$$

and hence that $(EF)=(\frac{1}{2})(AC)=(ED)$. In this case, therefore, F coincides with D and segment AH coincides with AD. The cut along ED is now unnecessary: to transform ACB into a square it is sufficient to cut the triangle just once along CD. Indeed, this fact is apparent without any computations. As Figure 5 shows, an isosceles right triangle is transformed into a square with the help of but one cut along the altitude to the hypotenuse.

Exercise 13 and its solution

We observed that Exercise 12 has application in the economical cutting of materials. The converse of Exercise 12 also has application in that direction.

Exercise 13: Given a square, by cutting it into at most four parts to transform it into a right triangle whose longer side is less than twice the shorter side.

Solution of Exercise 13: We take as a guide our solution of Exercise 12, and in particular Figure 4.

Let $AHKL$ be the given square, and let k be any fraction such that $1<k<2$. Our aim is to transform the square (with the help of three cuts) into a right triangle whose longer side is k times the shorter side.

Suppose this desired triangle has a shorter side of length a and a longer side of length b. Then

$$\frac{b}{a}=k \quad \text{and} \quad \frac{a\cdot b}{2}=(AH)^2.$$

If follows from these two equations that

$$(6)\quad a=(AH)\sqrt{\frac{2}{k}}=\sqrt{(AH)\cdot\frac{2}{k}\,(AH)}.$$

The last member of (6) tells us that a segment of length a can be obtained as the mean proportional between segment AH and a segment of length $2/k(AH)$. Suppose this segment of length a has been constructed.

On side HK of our given square mark a point C such that $(AC)=a$. Draw segment AC, and draw segment LM perpendicular to AC. (These constructions are illustrated in Figure 6.)

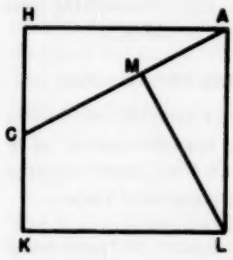


Figure 6

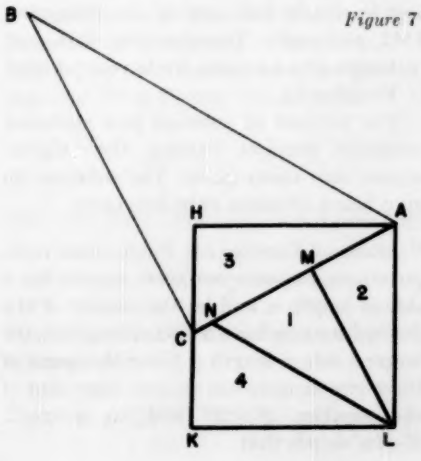
Now it is the case that the right triangle ACB with sides AC and BC, where $(BC)=2(LM)$, is equivalent to our original square $AHKL$. For the area of this triangle is $(\frac{1}{2})(AC)\cdot(BC)$, i.e., $(AC)\cdot(LM)$; and on the other hand, from the similarity of the triangles AHC and AML in Figure 6,

$$\frac{(LM)}{(AL)}=\frac{(AH)}{(AC)},$$

whence

$$(AC)\cdot(LM)=(AL)\cdot(AH)=(AH)^2.$$

Figure 7



The last stage of the solution sees us add to Figure 6, and obtain Figure 7, as follows: at C erect the perpendicular to AC, and on this perpendicular mark point B such that $(BC)=2(LM)$; draw AB; lastly, draw LN parallel to AB. Segments AC, LM, and LN divide square $AHKL$ into four parts which we labeled as shown in Figure 7. These four parts reassemble into the sought-for right triangle ABC.

Exercise 14 and its solution

Here and in the next exercise we concern ourselves with the transformation of a single square into two or more squares whose areas are in a prescribed ratio.

Exercise 14: Given a square, to transform it into two squares of which one has an area twice that of the other.

Solutions of this Exercise can be found in two different ways. One way is suggested by our second solution of Problem II, in the first (May, 1956) paper of this series. In connection with Problem II we learned how to transform a square into a rectangle whose longer side had a length three times that of the shorter side. Suppose we perform that transformation on our present square. From the resulting 1 by 3 rectangle, let us cut off a square with side congruent to the shorter side of the rectangle. This is one of the squares sought for; its area is clearly half that of the remaining 1×2 rectangle. Transforming this last rectangle into a square, we have a solution of Exercise 11.

The method of solution just sketched certainly requires cutting the original square into many parts. The solution we give below requires only five parts.

Solution of Exercise 14: First, some computations. Suppose our given square has a side of length a, and let the smaller of the desired squares have a side of length x, the larger a side of length y. Since the larger of these two squares has an area *twice* that of the smaller, $y^2=2x^2$ and so $y=x\sqrt{2}$. Hence we see that

$$x^2+(x\sqrt{2})^2=a^2;$$

and consequently

$$x=\frac{a\sqrt{3}}{3},$$

$$y=\frac{a\sqrt{6}}{3},$$

which is to say

$$x=\sqrt{a\cdot\frac{a}{3}},$$

$$y=\sqrt{a\cdot\frac{2a}{3}}.$$

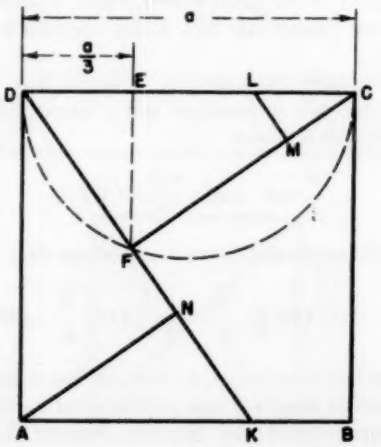
Thus x is the mean proportional between a and $a/3$, and y the mean proportional between a and $2a/3$.

These calculations form the basis of the construction now to be described. Let our given square be square $ABCD$ of Figure 8. With side DC as diameter, construct a semicircle inside $ABCD$. Now on DC mark the point E such that

$$(DE)=\frac{(DC)}{3}=\frac{a}{3},$$

and draw segment EF perpendicular to DC. Draw segments DF and CF. Recalling the mean proportional relations in the right triangle DFC, we see that

Figure 8



$$(DF) = x = \frac{a\sqrt{3}}{3},$$

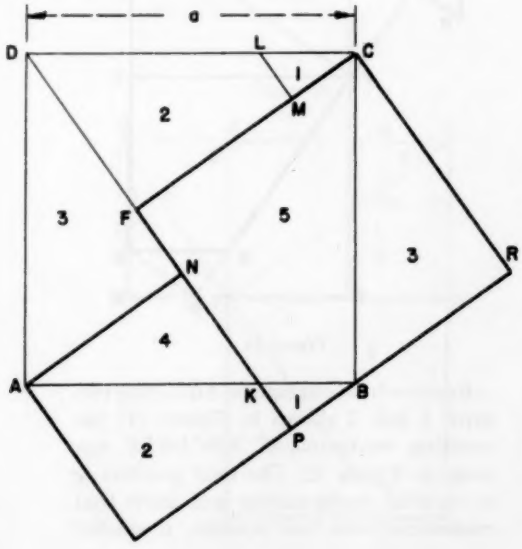
$$(CF) = y = \frac{a\sqrt{6}}{3}.$$

Next, extend DF to side AB and let K be the point of intersection of this extension with AB. On CD mark point L such that $(CL) = (BK)$, and draw LM perpendicular to CF. Finally, from vertex A draw the perpendicular AN to DK; notice that $(AN) = (FD)$, since the right triangles AND and CFD are congruent.

Cutting our original square $ABCD$ along the segments DF, AN, CF, and LM, we obtain five parts. From these five parts, as Figure 9 shows, we can readily assemble the two desired squares $CFPR$ and $ATPN$.

Comment on our solution of Exercise 14. On pp. 445–446 of our October 1956 paper we made some remarks about the converse of Problem III. These remarks suggest another way to solve Exercise 14. What is the connection between that method and our present one?

Figure 9



Exercise 15 and its solution

Exercise 15: Given a square, to transform it into three squares whose areas are in the ratio 2:3:4.

Solution of Exercise 15: Let our given square $ABCD$ have a side of length a. Of the three squares into which $ABCD$ is to be transformed, one is smallest, one is middle-size, one is largest. Suppose y is the length of the side of the middle-size square. The areas of the three squares sought are then respectively

$$\frac{2}{3}y^2,\ y^2,\ \frac{4}{3}y^2.$$

The sum of these three areas is $3y^2$, and we know that

$$3y^2 = a^2.$$

Easy calculations lead to the following results about the lengths x, y, z of the sides of the smallest square, the middle-size square and the largest square respectively:

$$x = \frac{a\sqrt{2}}{3},$$

$$y = \frac{a\sqrt{3}}{3},$$

$$z = \frac{2a}{3}.$$

Our solution of Exercise 14 showed how to construct a segment of length $a\sqrt{3}/3$. Using that construction here, we see that segment DF in Figure 10 is such that

$$(DF) = \frac{a\sqrt{3}}{3} = y.$$

Also, we know that $(DE) = a/3$ and that

$$(EC) = \frac{2a}{3} = z.$$

Finally, since DEF is a right triangle, it follows that

$$(EF) = \sqrt{(DF)^2 - (DE)^2} = \sqrt{\frac{a^2}{3} - \frac{a^2}{9}};$$

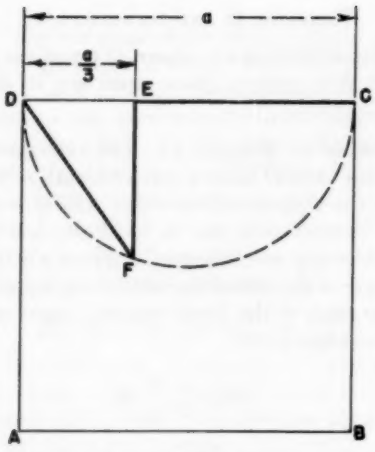


Figure 10

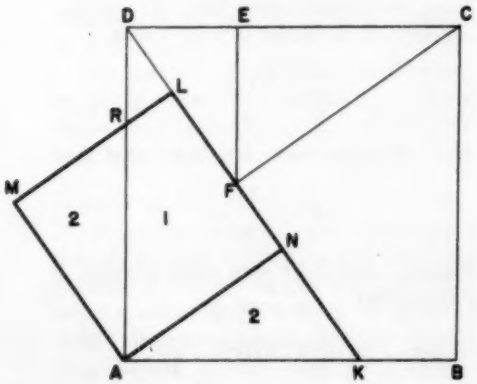
simplifying this radical, we see

$$(EF)=\frac{a\sqrt{2}}{3}=x.$$

We have constructed three segments EF, DF, EC of lengths x, y, z respectively.

Now add to Figure 10, and obtain Figure 11, as follows: extend DF to its intersection K on AB, draw AN perpendicular to DK, and construct the square $ANLM$ having AN for one side. Since triangles AND and DFC are congruent, it follows that $(AN)=(DF)=y$ and hence that *$ANLM$ is the middle-size one of the three squares sought.* This square $ANLM$ is composed of the trapezoid $ANLR$ (part 1 in Figure 11) and the triangle ANK (part 2 in Figure 11), since triangles ANK and AMR are congruent.

Figure 11



In connection with Figure 11 we note that congruence of triangles ANK and AMR implies $(AK)=(AR)$, and so

(7) $$(BK)=(DR).$$

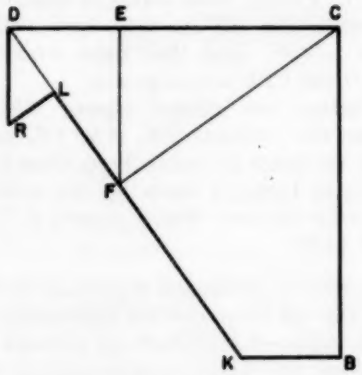


Figure 12

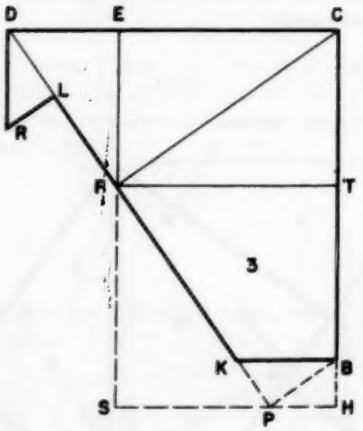


Figure 13

Remove from our square $ABCD$ the two parts 1 and 2 shown in Figure 11; the resulting configuration $KBCDRLK$ appears in Figure 12. The next problem is to cut this configuration into parts that reassemble into two squares, a smaller

one of side x and a larger one of side z. We proceed first to the larger square.

Add to Figure 12 and obtain Figure 13 as follows: draw segment FT perpendicular to BC (note that $(FT)=(EC)=z$), construct the square $FTHS$ having FT for one side, extend FK to its intersection P with SH, and draw segment PB.

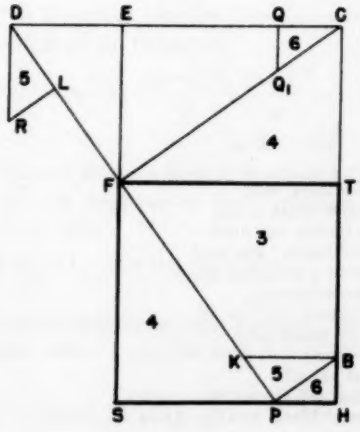
The trapezoid labeled '3' in Figure 13 is common to the square $FTHS$ and the original configuration $KBCDRLK$. Let us find other parts in the configuration which will fill out square $FTHS$.

Observe first that *triangles* FSP *and* FTC (both labeled '4' in Figure 14) *are congruent.* The reason: $(FS)=(FT)$ by construction; $<SFP=<TFC$ (since $<SFP+<PFT=90°=<TFC+<PFT$); and both $<PSF$ and $<FTC$ are right angles. In view of this congruence,

$$(SP)=(CT)=(EF)=\frac{a\sqrt{2}}{3}.$$

Next, we shall establish the congruence of the right triangle RLD in the configuration and the triangle KPB in square $FTHS$ (both triangles are labeled '5' in Figure 14). To this end we show first that KPB *is a right triangle.* Let us compute (PH) and (BH):

Figure 14



$$(PH)=(SH)-(SP)$$
$$=\frac{2a}{3}-\frac{a\sqrt{2}}{3}$$
$$=\frac{a\sqrt{2}}{3}(\sqrt{2}-1);$$
$$(BH)=(HT)+(TC)-(BC)$$
$$=\frac{2a}{3}+\frac{a\sqrt{2}}{3}-a$$
$$=\frac{a}{3}(\sqrt{2}-1).$$

From these calculations it appears that

$$\frac{(PH)}{(BH)}=\sqrt{2}.$$

Since earlier work tells us

$$\frac{(FT)}{(CT)}=\sqrt{2},$$

we now know that $<BPH=<CFT$. From this fact, and the fact that PH is parallel to FT, it follows that BP is parallel to CF. Consequently $<BPK$ is a right angle, and hence KPB is a right triangle. From this point we easily prove that *the right triangles* KPB *and* RLD *are congruent.* Proof: $(BK)=(DR)$ by (7); and $<KBP=<LDR$, since these are acute angles whose arms are perpendicular.

Finally, to fill out square $FTHS$ we still need to supply the right triangle BHP. Let us obtain this part by cutting it off triangle CEF: as shown in Figure 14, we simply mark on CE the point Q such that $(CQ)=(HP)$, and draw QQ_1 perpendicular to CQ. The two triangles labeled '6' in Figure 14 are thus congruent. We have obtained from the configuration of Figure 12 parts that assemble into a square whose side has length z; which is to say, *we have now obtained from our original square the largest of the three squares desired.*

Removing parts 3, 4, 5, 6, from the configuration of Figure 14, we are left with the quadrilateral $QDFQ_1$ shown in Figure 15. From this quadrilateral we must obtain

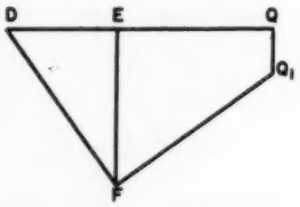
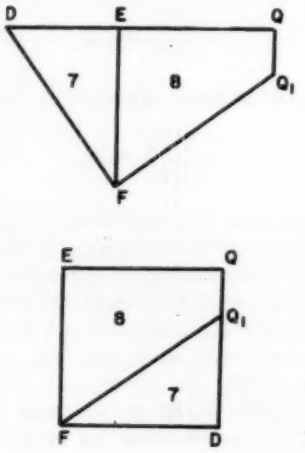


Figure 15

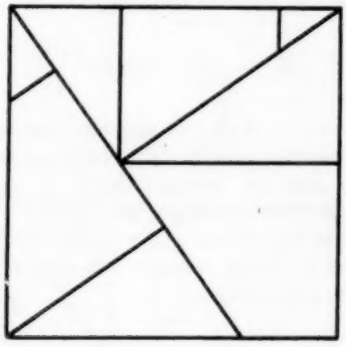
the smallest of the three desired squares, viz., a square of side x. It is easy to show that a single cut along EF divides the quadrilateral into two parts (labeled '7' and '8' in Figure 16) that reassemble into *the desired smallest square*. The solution of Exercise 15 is now complete.

Figure 16



Comment on the solution of Exercise 15. In Figure 17 we reproduce our original square $ABCD$ showing the eight parts into which our solution of Exercise 12 divided it. We know these eight parts reassemble into three squares whose areas are in the ratio 2:3:4. Here are other facts about these same eight parts: they can be reassembled into a single parallelogram; and they can be reassembled into two squares whose areas are in the ratio 1:2. It might amuse you to draw Figure 17 on light cardboard, cut out the eight parts, and solve these two "jigsaw" puzzles.

Figure 17



A mortifying mistake

I studied my tables over and over, and backward and forward, too;
But I couldn't remember six times nine, and I didn't know what to do,
Till my sister told me to play with my doll, and not to bother my head—
"If you call her 'Fifty-four' for a while, you'll learn it by heart," she said.
So I took my favorite, Mary Ann (though I thought 'twas a dreadful shame
To give such a perfectly lovely child such a perfectly horrid name),
And I called her my dear little "Fifty-four" a hundred times, till I knew
The answer of six times nine as well as the answer of two times two.
Next day Elizabeth Wigglesworth, who always acts so proud,
Said, "Six times nine is fifty-two," and I nearly laughed aloud!
But I wished I hadn't when teacher said, "Now Dorothy, tell if you can";
For I thought of my doll and—sakes alive!—I answered, "Mary Ann!"—*Anna M. Pratt*

The logarithm and its inverse

ROBERT C. YATES, *College of William and Mary, Williamsburg, Virginia.*

An approach to any topic that is somewhat "out of the ordinary" is a good thing. Secondary teachers may use this article to refresh their knowledge of calculus and gain new insight into ideas already familiar.

IT SEEMS IMPORTANT to define the elementary transcendental functions by means of power series before completing a first course in calculus. It is customarily made clear that these series are a real necessity, for example, in calculating function values for entry into tables. Perhaps it is not customary to display the fact that all properties of these functions may be derived directly from their series representations. This is the purpose of this paper. To sharpen the point, we shall discard prior knowledge, pretend ignorance of the function, and accept the series as its only characterization. The discussion here is limited to the logarithm and the exponential, which will parade under the labels L and E, respectively.

We must assume that the term-by-term differentiation of a series defining a function results in a new series which defines the derivative of the function within the same interval of convergence.

1. *The logarithm function.* The function L is defined by the series:

$$L(x)=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots, \quad \text{in } 0<x\leq 2 \quad (1)$$

wherein we note that $L(1)=0$ and that $L(x)$ is negative if $x<1$, positive if $x>1$. Now

$$L'(x)=1-(x-1)+(x-1)^2-\cdots, \quad 0<x<2$$

is recognizable as the *geometric series* with ratio $(1-x)$. Its sum is $1/x$. Thus

$$L'(x)=\frac{1}{x}$$

or in differential form:

$$\boxed{dL(x)=\frac{dx}{x}}. \quad (2)$$

2. *Properties of* L. We use the basic form (2) and positive arguments in the following:

(a) Since

$$dL(ax)=\frac{d(ax)}{ax}=\frac{dx}{x}=dL(x)$$

then

$$d[L(ax)-L(x)]=0,$$

or

$$L(ax)-L(x)=\text{constant}.$$

But for $x=1$, $L(1)=0$ and thus the constant has value $L(a)$. Accordingly,

$$\boxed{L(ax)=L(x)+L(a)}. \quad (3)$$

Here, terms of the right member are valid for $0<x\leq 2$ and $0<a\leq 2$. Thus the left member is validated for $0<ax\leq 4$. We are by such means permitted to extend the original interval in step-by-step fashion.

A corollary is established by setting $x=1/a$ in (3). We may write

$$L(1)=L\left(\frac{1}{a}\right)+L(a)$$

or

$$\boxed{L\left(\frac{1}{a}\right)=-L(a)}. \quad (4)$$

(b) Setting $x=1/b$ in (3) there follows

$$L\left(\frac{a}{b}\right)=L(a)+L\left(\frac{1}{b}\right)$$

or with (4):

$$\boxed{L\left(\frac{a}{b}\right)=L(a)-L(b)} \quad . \qquad (5)$$

(c) Writing (2) in the form $dL(u)=du/u$, we put $u=x^n$ where n is a rational number. Then

$$dL(x^n)=\frac{dx^n}{x^n}=\frac{nx^{n-1}dx}{x^n}=n\cdot\frac{dx}{x}$$

$$=n\cdot dL(x)$$

or

$$d\left[L(x^n)-n\cdot L(x)\right]=0.$$

Thus

$$L(x^n)-n\cdot L(x)=\text{constant}.$$

Putting $x=1$, the constant is evaluated as zero and accordingly

$$\boxed{L(x^n)=n\cdot L(x)} \quad . \qquad (6)$$

3. *The exponential function.* We now define a new function E by:

$$E(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots,$$

$$-\infty<x<+\infty \qquad (7)$$

and note that $E(0)=1$ and that $E(x)>0$ for $x\geq 0$. In addition

$$E'(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=E(x)$$

or

$$\boxed{dE(x)=E(x)dx} \quad . \qquad (8)$$

Thus

$$dx=\frac{dE(x)}{E(x)}=dL\left[E(x)\right]$$

or

$$d\left\{x-L\left[E(x)\right]\right\}=0.$$

Then

$$x-L\left[E(x)\right]=\text{constant}.$$

But for $x=0$, $E(0)=1$ and $L(1)=0$, and thus this constant is zero. We have then

$$x=L\left[E(x)\right],$$

an inverse relation; i.e., L and E are functions such that

$$L(E)=E(L)=I, \text{ the identity.}$$

4. *Properties of* E.

(a) If we let $y=L(x)$ and $k=L(a)$ so that $x=E(y)$ and $a=E(k)$, then

$$y+k=L(x)+L(a)=L(xa).$$

Taking the inverse of this expression, we find

$$\boxed{E(y+k)=xa=E(y)\cdot E(k)} \quad . \qquad (9)$$

(b) Setting $z=E(ax)$ so that $L(z)=ax$, then for rational a,

$$x=\frac{1}{a}\cdot L(z)=L(z^{1/a}) \text{ by (6).}$$

The inverse is

$$E(x)=z^{1/a}=\left[E(ax)\right]^{1/a}$$

or

$$\boxed{\left[E(x)\right]^a=E(ax)} \quad . \qquad (10)$$

Note that for $a=-1$, equation (10) yields

$$E(-x)=\frac{1}{E(x)}$$

which, in view of the remark following equation (7), discloses the fact that $E(x)$ is positive for all values of x. Furthermore, as $x\to\infty$, $E(x)\to\infty$ and $E(-x)\to 0$.

For the sake of familiarity, we list the last three forms in their usual garb:

$$dE(x)=E(x)dx \qquad de^x=e^x dx \quad (8)$$

$$E(y+k)=E(y)\cdot E(k) \qquad e^{y+k}=e^y\cdot e^k \quad (9)$$

$$\left[E(x)\right]^a=E(ax) \qquad (e^x)^a=e^{ax}. \quad (10)$$

Opportunities for careers in the mathematical sciences[1]

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Do your students know how varied are the opportunities for a career in mathematics?

WHAT ARE THE OPPORTUNITIES for careers in mathematics and the related mathematical sciences? What sort of attitudes should color the philosophy of young people contemplating a career in mathematics? What are some of the outstanding fields of endeavor in which young mathematics majors can find interesting, challenging, and rewarding careers? It is to such questions as these that I direct my attention today.

For many years the great majority of the serious students of mathematics have found their way into teaching positions, and those of them who have continued their studies leading toward the higher degrees have for the most part sought college or university teaching positions. The latter have followed the usual academic pursuits of teaching, and some of them with special interests have developed into research mathematicians interested in exploring mathematical problems.

This tradition of mathematics majors entering the teaching profession has undergone a sharp change, and in fact has been broken, during the last few years. Today, many mathematics majors plan their studies with the view of following a nonacademic career, though some of them so design their curricula that they will be equally well prepared for either an academic or an industrial career.

SELECTING A CAREER FOR SATISFACTION AND SERVICE

Many people labor under the impression that the prime aim in seeking a career in a particular field lies in the monetary compensation that may be expected. Naturally, compensation is indeed an important aim and must be given serious consideration. Of equal or even greater importance, however, are an enduring interest in a subject-matter field; aptitude and thorough training; and the personal satisfaction that can be derived from working in the field.

Associated with the compensation are the opportunities for advancement. While investigating opportunities for advancement one should realize that slow growth, often beset by real hurdles, and a requirement of high standards of performance, are in the end likely to be more rewarding than frequent promotions and a rapid acquisition of responsibility and authority. Real mature responsibilities can never be adequately and wisely exercised without a thorough apprenticeship and the development of mature competence and judgment. Too rapid a promotion commonly results in greater disappointments—both to employee and employer—than the slower well-earned rise in position.

[1] A lecture given to the Career Conference held at Knox College, Galesburg, Illinois, January 23, 1957.

[2] The views expressed herein are those of the author and are not necessarily those of the Navy Department.

Opportunities in mathematics

The opportunities for careers in mathematics have multiplied manyfold within the last fifteen years. Not only have universities and colleges continued to offer interesting careers in mathematics (and the number of such positions is on the increase), but in addition there has been a very rapid growth in the demand for and utilization of mathematical talents in industry, government, and research and development institutions.

Currently there are great demands for mathematically-trained people in practically every large industry in almost every section of this country. Our great industrial and government laboratories have under way much work of interest and importance, and this work creates a large and continuing need for the services of well-trained mathematicians. Careers in these fields of industry and government have much to offer.

For those interested in secondary education, the rapidly growing need for good mathematics teachers who know their subject offers a certain challenge and opportunity. If this nation is to continue to progress in a healthy manner we must improve the mathematical and basic scientific training offered to our younger people in our primary and secondary schools. The problems of our schools are serious and the opportunities for mathematically-trained teachers should grow appreciably, especially since certain weaknesses in this area are now acknowledged by many of the best minds in the country. A career in this field is just as important as a teaching career in a college or university, or a career in industry.

Research and development in universities, industry and government

The university scientist who devotes his time to research is usually engrossed in the frontier problems of his field. To him the search for the understanding of new phenomena is more absorbing than attempts to utilize his findings. He is not so apt to be concerned with developing the fundamentals of the technology of his fields. Thus, the typical research physicist in the university is apt to be more interested in the fundamental problems of subnuclear forces than he is in adapting the physical principles discovered in these fields to practical uses (e.g., as in reactors for generating electricity for commercial use).

The engineer in university research is apt to be concerned with developing new materials, new methods of design, new practices; or with acquiring a more comprehensive understanding of fatigue, vibration, or the thermodynamics of a possible new principle for an engine or motor. This attitude is in contrast with that of the practicing engineer, who is more likely to be engaged in actual design, construction, and operation of power distribution systems, structures—making use of the more conventional and better known types of technology. The university research engineer is thus a sort of intermediary between the pure scientist and the practicing engineer in industry.

Engineers, scientists, physicists—all can make much good use of mathematical techniques. Some of the mathematics they must use they will undoubtedly have the training to handle, but in many instances this will not be the case.

In many diverse fields of research activity, use of advanced mathematical techniques is urgently required. But who is going to create and develop the as-yet-undiscovered mathematical principles which can be drawn upon when the need arises? Who is going to develop such principles into suitable, well-knit mathematical structures that the engineer and scientist can lean upon for a better understanding, exposition, and prediction of the newer fields that he explores? This is the business of the competent research mathematician. This the university research mathematician has been doing for many years, often

generations ahead of the utilization of his research products. Such men have been very successful in bringing their talents to bear on many difficult problems. There is always opportunity for a brilliant career for the type of mathematician who can do this sort of research.

Who is going to have the background of knowledge of the mathematical sciences and their interplay with the sciences upon which the engineers, the scientists, the businessmen can draw for collaboration and help in the solution of their problems? It is the mathematician who is broadly trained, perhaps specifically in the fields of application involved. This mathematician need not be a university research mathematician. He may be a mathematician working in some industry or laboratory or business, who is one of a team of workers representing many fields of technology and knowledge. It is in this great field of service that so many opportunities and positions have opened during these last few years. Mathematicians who do this sort of work may not publish research papers like those of their academic brothers, but they may contribute much to the success of some industrial research, development, or product, and they may well be real investigators and research men of major accomplishments. Industry and government employ many mathematicians, and mathematicians have real opportunities to contribute to the success of new ventures and to enjoy a career of service and satisfaction.

Steps in the realization of a new product or system

In order to see more clearly just how mathematicians may fit into industrial or government positions it may be well to review some of the steps involved in the ultimate realization of a new product, or system. I have in mind such a thing as the development of a radically new type of propulsion system, or a new type of communication system, or the like. The realization of such a new system or product will more than likely be brought about by a multiplicity of efforts, any one of which (as well as the system as a whole) may involve the following general steps: pure research; applied research; development; construction of prototypes for study, testing, etc.; the redesign of prototypes for limited production; the limited production of the redesigned prototype, with subsequent evaluation, testing, etc.; the redesign for quantity production aimed at removing the weaknesses disclosed to date, and at the realization of a readily reproducible system of uniform and reliable characteristics; the actual production; storage; marketing; and, finally, the maintenance of the resulting system in actual mass use.

At several stages in the realization of the desired system, careful analysis and evaluation of the system are absolutely necessary to disclose the characteristics of the system, its utility, weaknesses, advantages, disadvantages, feasibility, reliability, and so on. Such analyses and evaluations commonly require the services of people of many talents. It has been found that mature mathematicians of sound judgment, widely versed in the physical, economic, and statistical sciences, are particularly well suited for this type of work.

All sorts of talents are made use of in such programs. In the pure research stage there may be no specific motive other than inherent curiosity, the search for new knowledge and understanding. Even later stages may involve an organized program of search. It is in these stages that the true university research methods of mathematics, physics, and chemistry should be expected to predominate. Later, when more of a specific phenomenon is known, there comes a stage when the problems of utilizing the discoveries become of real interest. This stage introduces many real challenges and commonly introduces a multitude of problems which fall naturally into the engineering-research type of activity. The gap between the discovery of a new physical phenomenon and the

actual utilization of the discovery may be a long and difficult one.

From this, one works into the formal development stage with all its trials and tribulations, tests, failures, etc. Eventually a prototype is evolved which can be made to operate with some sort of reliability. From the experiences with such prototypes there is evolved a design suitable for mass production and with some hope of achieving a uniformly dependable production product. Not least, of course, are the problems of marketing the system as produced and its subsequent successful operation and maintenance by people with little real knowledge of the system.

Various talents and degrees of training needed

Each of these steps leading to a satisfactory system requires various types of talents. Experts in the physical sciences are apt to predominate in the research and applied research stages, while the research engineer is apt to play a dominant role in the applied research and development stages, and the more practical type of engineer to take the helm in the development, prototype, design, and production stages.

The research mathematician may be expected to exert his principal efforts in the pure and, to some extent, in the applied research stages. The applied mathematician may in general find his services most needed in the research stage, but he may also be in great demand during the applied research, design, and development stages. His services may be used in the analysis and evaluation of the system at various stages in the process.

A mathematician trained to participate in a program leading to improved systems, such as is referred to above, may be required to work in many fields. At times his work may be far removed from the type of research which academic mathematicians in this country commonly think of as mathematical research. The services of such a mathematician in industry and government can nevertheless be of great value. He can bring to bear a tremendous background of knowledge and techniques in the arts of calculation and of analysis. Ideally, this would include mastery of numerical methods and the limitations and potentials of computing engines; the basic fields of applied mathematics (electrodynamics, electronics, elasticity, dynamics and statics, thermodynamics, heat conduction, etc.); the principal fields of analysis (ordinary and partial differential and integral equations, calculus of variations, real and complex analysis, analysis in general); the great fields of geometry (differential geometry, tensor analysis, and related fields); the mathematical theory and practice of probability and statistics (including a basic knowledge of information theory); the structure of modern abstract algebra (including some number theory); and modern mathematical physics (both classical and quantum). Obviously, such a background requires not only a good mathematical education, but, in addition, considerable knowledge of some of the more theoretical branches of engineering and physics, some of which can be acquired in the undergraduate school, some later.

Such a background is in many respects a formidable one. But a real applied mathematician should acquire it somehow. This does not mean, however, that all mathematical work done in industry or government is this exacting or advanced. Nor does this mean that everyone must master all of the things I have just listed. There is much room for an enterprising career for those who have majored in undergraduate mathematics and who have the will to learn new things and work hard.

One can begin a career in mathematics with merely an undergraduate major in mathematics. But if he is to progress and go very far in some types of activities he must continue his studies vigorously. One expecting to follow a career in college or university teaching, in research and de-

velopment, or in the more technical aspects of industrial or governmental work should pursue a well-planned course of study leading to an advanced professional degree or the Ph.D. degree. Such an effort should be made as soon after receiving the bachelor's degree as possible. This does not mean that all students who follow a career in mathematics should earn the doctorate degree, but it does mean that for the more advanced and difficult work, as much post-graduate training as is possible to get and assimilate properly should be sought. Different people have varying talents, interests, and aptitudes, and the extent to which the more advanced type of training should be pursued will depend largely on the individual's abilities, interests, and plans for a specific type of career.

Distinction between the points of view of the mathematician, the engineer, and the scientist

Perhaps the fundamental distinction between graduate mathematical training and the corresponding type of training in physics is that in the first what is important is the development of an understanding and an appreciation for the structure of the analytical work—the mathematics; whereas the emphasis in physics (even theoretical physics) is on the physical development of the theories. In border fields, either mode of approach alone is apt to leave one rather in the dark as to the underlying structure of the reasoning used, unless specific attention is given to development of the theme from both points of view.

The training for the physicist is concerned primarily with the preparation for a life of search leading to a better understanding of nuclear or subnuclear forces, and to the development of new theories which fit all the facts and permit accurate predictions of experiments—theories which reveal a broad quantitative understanding of a field of physics. The training for the engineer (especially one trained for advanced work and research) is designed to prepare the student primarily for work which will result in the development and realization of new practices, new materials, new methods of design, and better understandings of vibration and fatigue. Such training should fit one for work intermediate between the work of the pure scientist and the so-called industrial engineer. (The training for the industrial engineer is aimed principally at the development of fundamental knowledge necessary to carry out designs, perhaps based on more traditional technology.)

On the other hand, the training for the mathematician is concerned primarily with the development and structures of the various essential disciplines of mathematics. Thus a deep understanding of the underlying structure of abstract algebra—including group theory, invariant theory—will serve as an unusually valuable background for the understanding of various branches of physics; a well-rounded foundation in geometry will help one to grasp and interpret many applications to modern physics, too often presented as specific examples whose basic logical structure may not be understood even by the writer. For example, a good background of the algebra and differential geometry underlying the theory of tensors, spinors, and the representation of groups, aids very materially in the understanding and knitting-together of many current publications and efforts in the modern theories of the structure of physics.

In planning curricula for mathematics of the type described, it should be realized that techniques change so rapidly that it serves little purpose to spend too much time on specific details of some specialized branch (e.g., quality control vs. a broad foundation in probability and statistics). Instead, the effort should be placed on the fundamental structure of the subject matter.

The deeply trained mature mathematician of judgment, working in the fields of physical science and of engineering, can often bring to bear a perspective,

an over-all approach, a generalization, or a common architecture which can contribute much to the understanding, explanation, and knitting-together of a large class of phenomena. When it can be be made, his contribution is often of great significance.

Perhaps from this general discussion you will have gained some idea of the sort of careers one might follow in non-academic activities. The opportunities are great. A well trained mathematics major can advance just as far as his training, capabilities, and inclinations will permit. And his work can be very rewarding, just as it can be in the academic fields of teaching and research.

In planning a career it is very important today to cross subject lines: in making your plans for your life career, design your curricula to overlap into several fields of interest (e.g., the physical sciences, economics). This is very important even though you now plan to make mathematics your major. It is well to bear in mind that research and development in the world is not "topology" alone, or "numerical methods," or organic chemistry, or anatomy. It is, rather, a blending of any and all known disciplines into a homogeneous system designed to end up in something of better use to all men. To a great extent this is the work of all engaged in a professional career.

Concluding remarks

The opportunities for careers in mathematics are many and varied. They vary all the way from a career of service as a teacher of mathematics (be it in secondary, college, or postgraduate school) to a career as a mathematician in industry, business, government, or a research and development laboratory. Some careers may lead to research, some to development activities, and others to a career of service in an industry or business, as a member of a team helping to keep the wheels of progress running smoothly for the benefit of mankind.

Practically every industry in the country has opportunities for good men and women, trained in varying degrees in the mathematical and statistical sciences. You have for your selection such widely diversified activities as are to be found in the aircraft, communication, transportation, electrical, and heavy industries. You will find all sorts of opportunities in the various research and development laboratories (government and private) of this country. These opportunities may be in the realm of physical, medical, biological, economic, or social science. All of them make use of mathematics, and will make greater use of it in the days to come. In the areas of business as such you will find many opportunities, some of which are just now beginning to open up, for mathematically-inclined talent. Just how this latter field will develop remains to be seen, but good mathematical talent may well contribute to the charting of the course this activity may take.

Perhaps from these considerations you will find it easier to map your own career, be it what it may. If you do follow a career in mathematics, remember that the more you master it, and the wider you build your perspective into other worthy fields of knowledge and endeavor, the greater will be your understanding, progress, contribution, success, and happiness.

Because there is a demand for the publication of announcements of local interest, The Mathematics Teacher will print such items in the future if limited to 250 words or less.

Self-instructive enrichment topics for bright pupils in high school algebra

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Enrichment should entail content and activities that require a high order of creative imagination; it should deepen and strengthen understanding.

MEETING THE EDUCATIONAL needs of bright boys and girls is a problem of every school system. The problem is intensified in schools where large classes exist and where teachers have heavy assignments. Yet pupils of high ability must have an opportunity for the full development of their potentialities. This demand is clearly stated in the opening paragraph of the bulletin, *Education for the Talented in Mathematics and Science:*

> The public schools are dedicated to the task of providing opportunities for the maximum educational development of each pupil. If there is no discrimination, each student, from the slow learner to the rapid learner, from the most talented to the least talented, will have those educational experiences that will permit him to become of maximum worth to himself, to his community, and to society. In harmony with the purpose of education in a democracy is the important task of identifying and making provisions for the student with exceptional potential in each of the fields of learning.[1]

If a pupil is not challenged to make appropriate use of his time and abilities, he is likely to develop poor habits of study. Recognizing that an established pattern of poor study habits is likely to be carried into future work and even into life, it appears all the more important that the bright pupils be stimulated to do work which will help develop sound habits of concentrated study. At the same time the work should be of the kind that gives the pupils valuable mathematical experiences.

The bright pupil is receiving increasing attention in the United States. But the movement to make widespread provisions for bright students is barely past the stage of recognizing that it should be done. In mathematics, the familiar two-track, or multiple-track, offering provides some differentiation in opportunity for bright pupils. Seldom, however, do all the pupils who enroll in the conventional elective courses of algebra, geometry, and trigonometry have high aptitude for success in mathematics. Accordingly, it is seldom feasible to organize the courses for bright pupils exclusively. As to the extent of other facilities for bright pupils in mathematics, there is little experimental evidence. However, there is widespread agreement among authorities that the provisions for the bright pupil are woefully inadequate.

Enrichment of the school curriculum is widely endorsed by authorities, teachers, and school administrators as a method of providing for bright pupils. It is desirable from the viewpoint of administration, because it can be provided in special classes for bright pupils *or* in regular classes on an individual basis. The question naturally arises, "What is enrichment?"

[1] Kenneth Brown and Philip Johnson, *Education for the Talented in Mathematics and Science*, A Report of the Joint Conference of the Cooperative Committee on the Teaching of Science and Mathematics of the American Association for the Advancement of Science and the United States Office of Education (Washington, D. C.: United States Government Printing Office, 1953), p. 1.

The meaning of enrichment has been somewhat confused. Too often, enrichment has meant more problems of the same kind given the rest of the class and has not required of the bright pupil full use of his intellectual capacities. Sumpton, Norris, and Terman express their view on the nature of enrichment for bright pupils:

> This enrichment must be of a kind which will result in work done on higher levels of achievement, in challenges which in themselves might be discouraging to slower intellects, and in a release of the creative energy of the bright children.[2]

There is agreement among authorities that enrichment should involve modifications in both content and teaching methods. It should entail content and activities that require a high order of logical thinking and creative imagination; it should deepen and strengthen understandings.

There is considerable evidence that only a very limited amount of enrichment, in the sense described above, is being provided for bright pupils in mathematics. One of the main troubles has been that teachers do not feel capable of developing enrichment materials or do not have the time to do it. The task of locating and collecting appropriate enrichment materials and providing the needed individual guidance in their use is formidable, indeed. This paper is the report of a study whose primary purpose was to help teachers in this dilemma. The study dealt with the enrichment of algebra for bright pupils through the use of self-instructive topics.[3]

In view of the need for enrichment materials that high school mathematics teachers can use with a minimum of effort and time, the specific purposes of this study were: (1) to locate topics suitable for the enrichment of the usual content in first- and second-course high school algebra; (2) to develop these topics into written materials that bright pupils can use with little help from a busy teacher; (3) to use the enrichment materials experimentally during the 1954–55 school session; and (4) to develop means by which teachers may judge the effectiveness of such enrichment content.

A preliminary list of 30 topics that might be suitable for enriching high school algebra for bright pupils was made. The writer was guided by the following criteria in preparing the preliminary list: (1) the topics should be primarily algebraic; (2) the list should not include the usual topics found in the standard first- or second-course algebra; (3) the topics should teach some mathematics and not be only for recreation or enjoyment. This list of topics, with a brief description of the proposed content of each, was sent to twelve men considered to be experts in mathematics education with the request that they check each topic as suitable or unsuitable for enriching high school algebra. Of the 30 topics, 27 were deemed suitable for enriching high school algebra.

The topics were written to be used by the pupil with little or no help from his teacher. Throughout each topic, explanations were made and followed by a set of exercises for the pupil to work. It was considered important that the pupil make both inductive and deductive algebraic generalizations. Opportunities for generalizing were provided frequently by leading the pupil through a series of questions and/or exercises to the desired generalization. A topic mastery test was prepared for each topic. Answers to all exercises and mastery tests were provided in separate booklets. To obtain the reactions of pupils who used the topics and of their teachers, separate "Reaction Sheets" for pupils and teachers were prepared.

Of the 27 topics deemed suitable by the panel of experts for enriching high school algebra, 24 were developed into written materials for use by bright pupils. Some topics were complete in themselves, while others formed a sequential series.

[2] Merle Sumpton, Dorothy Norris, and Lewis Terman, "Special Education of the Gifted," *Forty-Ninth Yearbook of the National Society for the Study of Education*, Part II (Bloomington, Ill.: Public School Publishing Co., 1950), p. 267.

[3] Joseph Payne, "Enrichment Topics For First and Second Course Algebra for Bright Pupils," unpublished doctoral dissertation, University of Virginia, 1955.

Topics developed that were complete in themselves are listed below with a brief description of each:

1. *Algebra in Retrospect.* Notes from the history of algebra.
2. *Approximate Numbers.* Nature of approximate numbers and their use in computation; meaning of precision, accuracy, error, and relative error of measurement as well as the use of significant digits as a guide for determining the accuracy of measurement.
3. *Congruence of Numbers.* Meaning of congruence of numbers; fundamental operations of addition, multiplication, subtraction; development of rules of divisibility by 2, 3, 4, 5, 8, 9, and 10.
4. *Cube Roots of a Number.* Using algebra to find the three cube roots of a number.
5. *Domain and Range of a Function.* Definition of domain and range and the relation of one to the other.
6. *Instruments for Indirect Measurement.* Building and using the clinometer, hypsometer, a simple mirror device, and the carpenter's square: meaning of indirect measurements and directions for making them.
7. *Magic Squares.* The general method of building a magic square of order three by algebraic deduction.
8. *Mathematical Recreations and Puzzle Problems.* Number problems and games, problems of reapportionment by decanting, and puzzle problems, including algebraic explanations when appropriate.
9. *Men of Mathematics.* Short biographies of Pythagoras, Euclid, Al-Khowarizmi, Leonardo of Pisa, Vieta, Napier, Newton, and Gauss.
10. *Number Systems Other Than Ten.* A study of the number systems with bases of 5, 12, and 2, including operations in each system; writing numbers as polynomials; equations and written problems in a base-12 system.
11. *Postulational Algebra.* A short introductory study of algebra through a postulational approach where each statement made is supported by a postulate, definition, or proved generalization.
12. *Prime Numbers.* Definition of prime number; use of Sieve of Eratosthenes for finding prime numbers, Fermat's formula, and other expressions that yield prime numbers, leading to a statement of the infinitude of primes.
13. *Pythagorean Relationship.* Finding integral solutions to the equation $a^2+b^2=c^2$, including the development of formulas for finding integral solutions.
14. *Rate of Change.* Basic ideas of constant change and rate of change of a function and of statistical data, using logarithmic graph paper to picture rate of change in different sets of data; emphasis on rate of change to statistical data.
15. *Scientific Notation.* Expressing any number as product of two numbers, one between 1 and 10 and the other a power of 10; numerical computations using scientific notation.
16. *Short Cuts in Computation.* Algebraic development of rules for multiplying and dividing by 10 and powers of 10, 25, 50, 75, and 125; algebraic explanation of rule for squaring a number ending in 5. .
17. *Slide Rule.* Instructions for building and using a slide rule; the use of logarithms to understand the slide rule and its operation.

There were two sets of sequential topics. Three topics comprised one sequential set:

18. *Arithmetic Mean.* Calculation of arithmetic mean by the short method including the mean from a frequency distribution; calculation of mid-mean.
19. *Mean and Standard Deviation.* Calculations of mean and standard deviation and their use to measure variation within a set of numbers; use of standard deviation to make simple probability calculations.

20. *Normal Curve of Probability.* Mathematical definition and properties; tables for ordinate and area of normal curve; use of area table to calculate probabilities from data which fit the normal curve.

The second sequential set contained four topics:

21. *Permutations.* Permutations of objects, none of which are alike and some of which are alike; circular permutations.
22. *Combinations.* Number of combinations from given number of objects; intuitive derivation of formula for finding number of combinations; proof that ${}_nC_r = {}_nC_{n-r}$.
23. *Binomial Expansion.* The expansion of $(a+b)^n$; the derivation and use of Pascal's triangle for finding the coefficients; writing coefficients of binomial expansion as combinations.
24. *Probability.* Definition and calculation of theoretical and experimental probability; relation of coefficients of binomial expansion to probability; calculating probabilities of more than one event at a time; probability as related to life insurance; graphing the probability distribution.

The enrichment topics were used experimentally at Indian Springs School, an independent secondary school for boys located near Birmingham, Alabama, and by six mathematics teachers in public high schools in Alabama, Tennessee, and Florida. Eighty-four pupils completed one or more enrichment topics. The 84 pupils completed a total of 209 enrichment topics. For the 77 pupils for whom I.Q.'s were available, the median I.Q. was 123.

The manner in which the topics were used was left largely to the teacher. The teachers, in the set of instructions, were given these possible ways for the enrichment topics to be used:

1. A student finishes a unit of work in the regular algebra course before the class does and then chooses an enrichment topic while waiting for the class to finish the unit.
2. A student may be doing his regular algebra work along with the class, but he has extra time to study an enrichment topic with his regular class work.
3. A teacher works with a small select group of students on a particular enrichment topic as often as she can spare the time from her regular class work.
4. Students use the enrichment topics as individual or group projects for a mathematics club.

The topics were used largely on an individual basis. A pupil chose a topic for study and worked on it as he found time, either in class or outside of class. No reduction was made in the class requirements for pupils who were working on enrichment topics, and no "extra credit" was given pupils who completed topics. The pupils were told that the prime purposes of the enrichment topics were to further their own knowledge and to broaden and deepen their mathematical background.

Some of the conclusions of the study follow:

1. There appeared to be a close positive relationship between the mathematical ability of pupils and their success in the study of enrichment topics. On the other hand, the relationship between intelligence quotients and success in the study of enrichment topics did not appear to be so high.
2. Many bright pupils found time to study enrichment topics and at the same time maintained a high standard of work in their regular mathematics course.
3. The enrichment topics seemed to produce a more enthusiastic pupil attitude towards mathematics.
4. In general, the bright pupils in mathematics were more interested in the mathematical ideas than in the computation of the answer to a problem. Also, they tended to minimize the

importance of accuracy in computation.

5. Pupils seemed more interested in studying topics that were complete within themselves than they were in following a series of sequential topics.
6. Content that was new and novel helped to motivate pupils in their study.
7. For very bright mathematics pupils, who are also active and curious, it seemed hardly possible to enrich too much. The pupils who met with greatest success in the study of enrichment topics showed no indication of being overworked or of reaching a ceiling in their learning.
8. The extent to which enrichment topics were used by pupils appeared to depend in a large measure on the enthusiasm of the teacher for the topics and his ability to motivate pupils to undertake topics for study.
9. All teachers using the topics said that the use of the enrichment topics in their classes required practically none of their time. This conclusion is of particular significance because almost all of them stated that they were overworked and had little free time.
10. Pupils tended to finish a topic sooner when a definite time-allotment was suggested. This was particularly true when a pupil first began the study of enrichment topics.
11. As a rule, pupils were able to follow the written explanations in the topics with comparative ease.
12. In writing the materials contained in the enrichment topics, the writer did not include extensive exercises. It was found that the number of exercises were sufficient for bright pupils to understand the materials and also to do well on the topic-mastery tests.
13. While the enrichment of algebra was the primary concern of this study, sufficient use of the topics was made by geometry pupils to indicate that pupils with the necessary general mathematical maturity will study enrichment materials that do not relate directly to the subject currently studied.
14. There was evidence that teachers participating in this study were making no special provisions for bright pupils in their classes. Use of the enrichment topics, in most cases, was their first attempt to make special provision for such pupils.
15. This study has clearly shown that bright pupils can do considerably more than they are presently required to do. The idea that bright pupils are often not fully challenged has received additional experimental evidence.

"Mathematics is the art of thinking with the aid of a pencil. It is neither deep nor profound, and is difficult only in the sense that some of the work is rather detailed and tedious."

Editorial Note: Found in a college publication. Is it possible to find a more inaccurate statement?

A note on number creation

CHARLES BRUMFIEL, *Ball State Teachers College, Muncie, Indiana.*

The author explores some common misconceptions about number and creates a new number system.

IT IS NOT UNUSUAL for a student in beginning college algebra to remark that the number 2 has no square root. If pressed to explain his statement, he may hedge by stating that even though a square root of 2 may exist, yet it is impossible to find it exactly.

The failure to understand what is involved in the creation of irrational numbers has probably much to do with the difficulties students encounter in learning to calculate with irrationals. Many students acquire a new insight into the structure of the number system of algebra when they are shown a method for constructing a number whose square is *exactly* two.

The first remark that needs to be made to the student who thinks we can not find a square root of two is that man does not find numbers; he creates numbers. Man has built number concepts with his mind just as surely as he built the pyramids with his hands. It is quite true that if one looks among particular sets of numbers one may fail to find a number with some prescribed property. For example:

1. If we look among the integers, we find no number which satisfies the equation $2x=1$.
2. If we look among the integers, we find no number of the form a^2+3 which is exactly divisible by 8.
3. Looking among the odd integers, no one has yet been able to find a perfect number, that is, an integer equal to the sum of its proper divisors (6 and 28 are perfect even numbers).
4. If one looks among the 999,999,999 consecutive integers following the integer 1,000,000,000!+1, one finds no prime number.
5. It is quite true that if one looks among the fractions (rational numbers) he will find no number whose square is 2.

Several interesting proofs of the last statement above have been given. A simple intuitive proof, which the author has seen spontaneously invented by various college freshmen, is the following:

Let m/n be any fraction reduced to lowest terms. Now it is impossible for $(m/n)^2$ to equal 2. For if $m \cdot m/n \cdot n = 2$, then every prime factor of the denominator must divide the numerator. But the only prime factors of the denominator are divisors of n, and none of these will divide the numerator.

Of course this proof takes for granted certain basic theorems from number theory. More elementary proofs may be presented. If one thinks of the relatively prime integers m and n as written in binary notation, then either the final digits for both m and n are 1, or one has final digit 1 and the other has for final digit 0. Now it is easy to see that in neither of these cases can we have $m^2=2n^2$. A proof similar to this last can also be given by considering the ordinary decimal representations of m and n and showing that it is impossible for the final digits of m^2 and $2n^2$ to be identical, if m and n are relatively prime.

Students are aware of the fact that in order to have a solution for the equation $2x=1$ man created the number $\frac{1}{2}$. They do not see quite so clearly that in order to have solutions for the equation $x^2=2$ man created the numbers $\sqrt{2}$ and $-\sqrt{2}$. A few sceptics may even ask, "What right have you to arbitrarily create a new number and claim that its square is 2?" The following exposition has proved illuminating.

We invent a new number system whose elements are ordered pairs of integers. For example, (1,2), (0,3), (2,5), $\cdots$ are numbers in this new system. We define sum and product for any two of these numbers by the rules,

$$(a,b)+(c,d)=(a+c,\ b+d)$$
$$(a,b)\cdot(c,d)=(ac+2bd,\ ad+bc).$$

According to these rules,

$$(1,2)+(3,4)=(4,6)$$
$$(-1,2)\cdot(-3,4)=(19,-10).$$

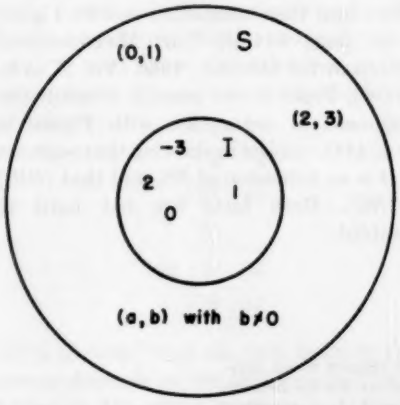
We observe that this new number system contains an interesting subsystem, namely those numbers of the form $(a,0)$. For these numbers we have, by our rules,

$$(2,0)+(3,0)=(5,0);\ (2,0)\cdot(3,0)=(6,0).$$

And in general,

$$(a,0)+(b,0)=(a+b,\ 0);$$
$$(a,0)\cdot(b,0)=(ab,\ 0).$$

In other words, the marks "$(a,0)$" and "$(b,0)$" may be considered to be nothing but new names for the integers a and b. Hence we identify $(a,0)$ with a and replace the number pairs having this particular form by the integers themselves. We have now a large number system S which contains the integers I as a subsystem. Symbolically,



Our calculations in S which involve the integers can be effected by replacing any integer a by the number $(a,0)$ and calculating according to our rules. For example,

$$2+(3,4)=(2,0)+(3,4)=(5,4)$$
$$3\cdot(2,-1)=(3,0)\cdot(2,-1)=(6,-3)$$
$$(2,3)+(3,-3)=(5,0)=5$$
$$(2,3)\cdot(2,-3)=(-14,0)=-14$$

Now we can prove the important theorem.

Theorem: There are two numbers in S, *namely* (0,1) *and* (0,−1), *whose squares equal* 2.

The proof is entirely computational and is left to the reader. We observe that these numbers are properly called the negatives of each other since

$$(0,1)+(0,-1)=(0,0)=0.$$

If we choose one of these two numbers, say $(0,-1)$, and agree to represent it by the new symbol "$\sqrt{2}$," we see that we may write

$$(a,b)=(a,0)+(0,b)=a+(-b)\cdot(0,-1)$$
$$=a+(-b)\sqrt{2}.$$

Now, clearly, every number of the set S can be represented in the form $r+s\sqrt{2}$ where r and s are integers. We observe that $(x,y)=x-y\sqrt{2}$ and $(x,-y)=x+y\sqrt{2}$.

The above procedure answers completely the question, "How can one construct a number whose square is 2?" The reader will observe that this large system S which contains the integers has been constructed from the integers much as a building might be constructed from bricks. The only completely honest way to bring a new number into existence is to construct it out of the old numbers which one has at hand.

The reader might find it interesting to prove the following.

Theorem: In the system S *there is no number* x *such that* $x^2=3$.

An excellent exercise in the methods of this article would be to attempt to construct, in a similar fashion, a mathematical system T which would contain S as a subsystem and in addition would contain an element t such that $t^2=3$.

An observation on dissecting the square

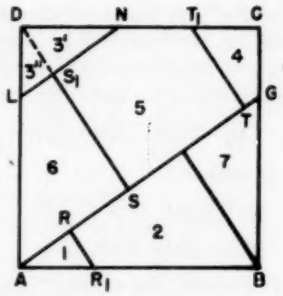
CHESTER W. HAWLEY, *Southwest Miami High School, Miami, Florida.*

This article tells how one teacher made use of the articles on dissecting the square published in last year's MATHEMATICS TEACHER

IN CONNECTION WITH THE RECENT series of interesting articles on dissection of figures,[1] I have an observation of possible value to those who, like myself, made plywood cutouts of the dissections for classroom use. The observation is this: *By a single additional cut, the given dissection of a square into three congruent squares can also furnish a dissection of the square into two squares.*

Refer to Figure 7 on page 340 of THE MATHEMATICS TEACHER for May, 1956 (Vol. XLIX, No. 5). This figure shows cutting lines for dissecting square $ABCD$ into parts that reassemble into three congruent squares. Suppose now we add to Figure 7 the segment S_1D, and thereby dissect triangle DLN into two parts, $3'$ and $3''$ respectively.

The extra cut S_1D does not, of course, spoil the intended outcome of Figure 7:



simply seal up S_1D again, and the resulting seven parts of $ABCD$ reassemble into three congruent squares. On the other hand, if S_1D is cut and cuts LN and T_1T are sealed up, the resulting five parts of $ABCD$ reassemble into two squares.

Evidence in support of this remark may be obtained by turning over the figure above and then comparing it with Figure 5 on page 444 of THE MATHEMATICS TEACHER for October, 1956 (Vol. XLIX, No. 6). Proof of the remark (consult the argument in connection with Figure 5, page 444) consists in showing that segment S_1D is an extension of SS_1 and that $(BR_1) = (BG)$. Both facts are not hard to establish.

[1] The five articles of this series all appeared in THE MATHEMATICS TEACHER during the last eighteen months: "A Problem on the Cutting of Squares," XLIX (May, 1956), 332–346; "More on the Cutting of Squares," XLIX (October, 1956), 442–454; "Still More on the Cutting of Squares," XLIX (December, 1956), 585–596; "New Exercises in Plane Geometry," L (February, 1957), 125–135; and "More New Exercises in Plane Geometry," L (May, 1957), 330–339.

"What education has to impart is an intimate sense for the power of ideas, for the beauty of ideas, and for the structure of ideas, together with a particular body of knowledge which has peculiar reference to the life of the being possessing it."—*Alfred North Whitehead.*

Degree of accuracy of square roots

C. STANLEY OGILVY, *Hamilton College, Clinton, New York.*

The author reminds us that it doesn't pay to generalize hastily

AN ODD MISCONCEPTION has crept into some textbooks and thence into the classroom, namely, that the process of extracting arithmetic square root yields a result which has only half as many significant figures as the original number. A specific example from a current book will perhaps recall for the reader similar statements made by his instructors in years gone by. The square root of 27.36 is calculated in the usual fashion to be 5.2, whereupon the author comments that "we are not justified in obtaining an answer beyond 5.2 as the nearest square root." We choose this sample because it comes not from some forgotten schoolbook of the nineteenth century but from the "arithmetic review" section of a college textbook published in 1952 by a highly reputable company which takes pride in the general excellence and accuracy of its texts.

The fact is that always three and essentially four figures in the answer are justified in an example like the above. It is not hard to see where the mistaken notion comes from. The arithmetic is displayed as follows:

$$\begin{array}{r|l} & 5 \cdot 2 \\ \sqrt{27} & \overline{36} \\ 25 & \\ \hline 102 \mid & 2\ 36 \\ & 2\ 04 \end{array}$$

"It is obvious" that the next figure in the answer depends on the next two (missing) figures in the given number, and hence that we cannot safely supply a pair of zeros after the 27.36. This is just another example of an "obvious" thing that isn't so. In long division, it is so; and the mind jumps over-zealously to promote a false analogy. Let us investigate the general situation and then return to a further consideration of this example.

The method of approximation by differentials is helpful. If $y=x^{1/2}$, $dy=\frac{1}{2}x^{-1/2}dx$. Let N be the given number whose square root is to be taken. Let x be the same number shorn of the decimal point and any terminal zeros. (For instance, if $N=27.36$ or 273600, then $x=2736$.) Take first the case where the decimal point has been moved an even number of places: $x=100^kN$, k a positive or negative integer or zero. Then $y=\sqrt{x}=10^k\sqrt{N}$, $dx\leq\frac{1}{2}$, and $dy\leq 1/(4\sqrt{x})$. We can tabulate dy as follows:

If $10^0\leq x<10^1$, $\sqrt{x}\geq 1$, $dy\leq .25$ <1.0
(first significant figure)

If $10^1\leq x<10^2$, $\sqrt{x}>3$, dy $<.1$
(second significant figure)

If $10^2\leq x<10^3$, $\sqrt{x}\geq 10$, $dy\leq .0025<.01$
(third significant figure)

If $10^3\leq x<10^4$, $\sqrt{x}>30$, dy $<.001$
(fourth significant figure)

—and so on.

It is evident that these inequalities could be considerably sharpened; but it is enough for our purposes to note that in no case can the square root of a number having m significant figures contain an error of more than unity in the m'th significant figure.

If the decimal point has to be moved an

odd number of places in the transition from N to x, we can make the substitution $z = 10x$. Then $dz = 10dx$, and dy is increased by a factor of $\sqrt{10}$. But of course this factor is divided out again in the final return from $\sqrt{10x}$ to $\sqrt{x}$, and we are left with the same result as before.

Returning now to the original textbook example, let us assume that 27.36 has been rounded off from some longer decimal. Then the "exact" value must lie between 27.355 and 27.365. But

$$\sqrt{27.35\ 50\ 00\ 00} = 5.2302, \text{ and}$$

$$\sqrt{27.36\ 50\ 00\ 00} = 5.2312.$$

Thus we see that regardless of what the exact value was, the round-off does not affect the square root until the fourth significant figure. Furthermore, if we calculate $\sqrt{27.36\ 00\ 00\ 00}$ from scratch (not interpolating between the other two), we obtain, of course, 5.2307, the mid-value. Rounding off to 5.231 introduces an error of at most one in the fourth figure, as predicted by the theory.

In general, then, except for a possible error of ± 1 in the last significant figure, one obtains a square root accurate not to half as many but to the same number of significant figures as were given in the original number.

For an extended treatment of the maximum error to be expected in extracting not only square roots but also cube, fourth, fifth, and sixth roots, the interested reader is referred to Section 14 of *Introduction to Numerical Analysis*, F. B. Hildebrand (McGraw-Hill Book Co., Inc., 1956).

What's new?

BOOKS

Secondary

Improving the Arithmetic Program, Leo J. Brueckner. New York: Appleton-Century-Crofts, Inc., 1957. Paper, vii+120 pp., $1.25.

Preparatory Mathematics I, Outlines and Exercises, Joel S. Georges, Theodore S. Sunko, Milton D. Eulenberg, and Harold M. Piety. Ann Arbor: J. W. Edwards, Publisher, Inc., 1957. Paper, 208 pp., $4.00.

Trigonometry, Rolland R. Smith and Paul P. Hanson. Yonkers: World Book Company, 1957. Cloth, x+470 pp., $3.72.

College

An Introduction to Algebraic Topology, Andrew H. Wallace. New York: Pergamon Press, 1957. Cloth, vii+198 pp., $6.50.

College Algebra, Irwin K. Feinstein and Kenneth H. Murphy. Ames, Iowa: Littlefield, Adams and Company, 1957. Paper, xi+362 pp., $1.75.

Elementary Mechanics, D. A. Quadling and A. R. D. Ramsay. London: G. Bell and Sons, Ltd., 1957. Cloth, viii+272 pp., 14s.

Miscellaneous

Archimedes, E. J. Dijksterhuis. New York: The Humanities Press, 1957. Cloth, 421 pp., $12.50.

The Fascination of Numbers, W. J. Reichmann, Fair Lawn, New Jersey: Essential Books, Inc., 1957. Cloth, 176 pp., $4.00.

Reason and Chance in Scientific Discovery, R. Taton (translated by A. J. Pomerans). New York: Philosophical Library, 1957. Cloth, 171 pp., $10.00.

BOOKLETS

Paper Folding for the Mathematics Class, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C. 36-page illustrated booklet written by Donovan A. Johnson, 75¢ each.

A new look at medians

ISRAEL KORAL, *Boys High School, Brooklyn, New York.*

Mathematics teachers always like new proofs for old theorems.

THE USUAL PROOF of the concurrency of the medians of a triangle depends on the theorems of Book I, and involves the auxiliary construction of a line joining the midpoints of the two longer segments of two intersecting medians.

A more elegant proof can be developed by making use of the proposition that the areas of two triangles with equal altitudes are to each other as their bases, as in the proof that follows. (Of course, the concurrency follows from the application of the proposition proven below to the third median's intersection of the first, EB.)

Proposition:

In a triangle the segments of a median formed by the intersection of another median are in the ratio 2:1. (See Figure.)

Given:

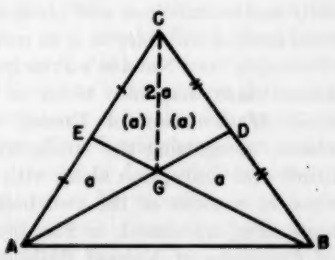
$\triangle ABC$. AD and BE are medians, intersecting at G.

To Prove:

$$BG:GE = 2:1$$

Proof:

1. $\triangle ABE = \triangle ABD$ (Since each is half the $\triangle ABC$, having the same altitude and half the base)
2. $\therefore \triangle AGE = \triangle BGD$ (Subtracting $\triangle AGB$ from the triangles in step 1)
3. $\triangle CGE = a$ (Equal bases, same altitude)
   $\triangle CGD = a$ (a = area of $\triangle AGE$)
4. $\triangle BGC = 2a$
5. $\triangle BGC : \triangle CGE = 2a : a = 2:1$
6. $\triangle BGC : \triangle CGE = BG:GE$ (Triangles having = altitudes are to each other as their bases.)
7. $\therefore BG:GE = 2:1$



The ability to imagine relations is one of the most indispensable conditions of all precise thinking. No subject can be named, in the investigation of which it is not imperatively needed; but it can be nowhere else so thoroughly acquired as in the study of mathematics.—*John Fiske.*

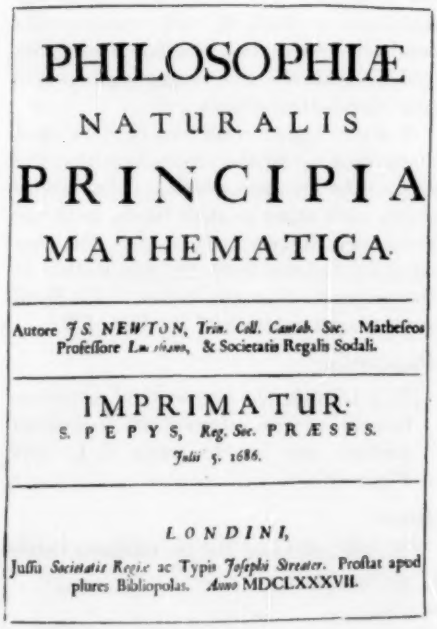
• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Sir Isaac Newton: 1642-1727

Isaac Newton has received so many honors and has been the subject of so many biographies both during his lifetime and in the two hundred and thirty years since his death that such a note as we can write here seems hardly necessary.[1] Yet the successful launching of an earth satellite owes so much to his genius that it seems appropriate again to call him to the attention of mathematics teachers and students and to display some mementos which may typify his genius.

Figure 1 shows the title page of one of the most famous books of the western world. Its fame is such that scientists, especially mathematicians and physicists, the world around will refer to it as merely "The Principia" or "Newton's Principia." (Mathematicians may also think of the *Principia Mathematica* of Russell and Whitehead, recognizing the similarity in magnitude and conception along with the difference in content of the two books.) Newton's title, translated, is *The Mathematical Principles of Natural Philosophy.*



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore *J S. NEWTON, Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater.* Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

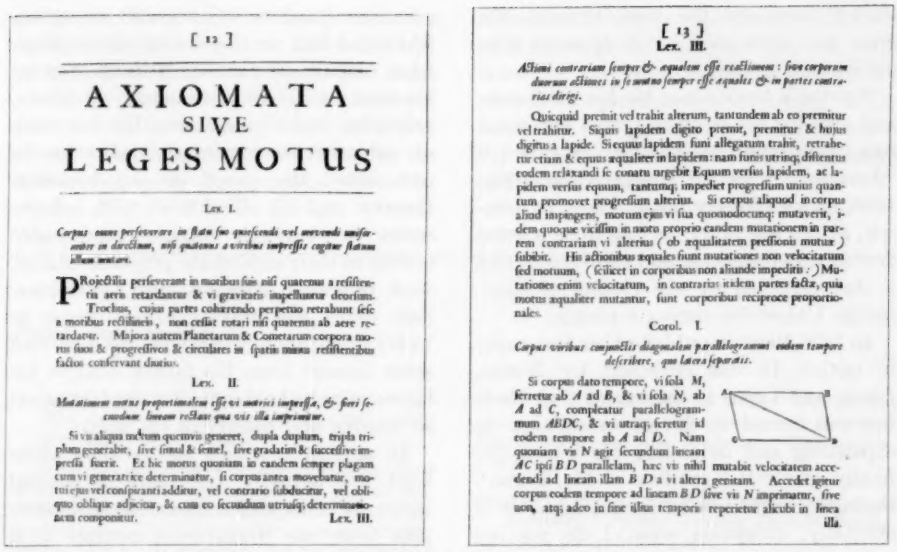
Figure 1

At the time it was first published "natural philosophy" meant science in general and physics and astronomy in particular.

Figure 2 shows the three "axioms" out of which Newton developed his mechanics. These simply stated, broad generalizations represent three major but closely related aspects of Newton's genius: (1) he generalized and formulated fundamental relationships which are the common unifying elements of apparently diverse phenomena; (2) he verified his generalizations

[1] The following books have been largely used in preparing this note. Detailed references will, in general, be omitted. E. N. da C. Andrade, *Sir Isaac Newton* (London: Collins, 1954). E. T. Bell, "On the Seashore," *Men of Mathematics* (New York: Simon and Schuster, 1937), pp. 90–116. This is also now available in a reprinted edition. History of Science Society, *Sir Isaac Newton, 1727–1927* (Baltimore: The Williams and Wilkins Co., 1928). H. W. N. Sullivan, *Isaac Newton, 1642–1727* (New York: Macmillan, 1938). W. W. Turnbull, *The Mathematical Discoveries of Newton* (London: Blackie and Son, Ltd., 1945).

Since it is recently published and has much of value in it, attention should be called here to the three articles on Newton reprinted in volume I of *The World of Mathematics* edited by James R. Newman (New York: Simon and Schuster, 1956).



[12]

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Projectilia perseverant in motibus suis nisi quatenus a resistentia aeris retardantur & vi gravitatis impelluntur deorsum. Trochus, cujus partes cohærendo perpetuo retrahunt sese a motibus rectilineis, non cessat rotari nisi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus suos & progressivos & circulares in spatiis minus resistentibus facilius conservant diutius.

Lex. II.

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, sive simul & semel, sive gradatim & successive impressa fuerit. Et hic motus quoniam in eandem semper plagam cum vi generatrice determinatur, si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel obliquo obliquæ adjicitur, & cum eo secundum utriusq; determinationem componitur.

Lex. III.

[13]
Lex. III.

Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus æqualiter in lapidem: nam funis utrinq; distentus eodem relaxandi se conatu urgebit Equum versus lapidem, ac lapidem versus equum, tantumq; impediet progressum unius quantum promovet progressum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi sua quomodocunq; mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem pressionis mutuæ) subibit. His actionibus æquales fiunt mutationes non velocitatum sed motuum, (scilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, sunt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore describere, quo latera separatis.

Si corpus dato tempore, vi sola *M*, ferretur ab *A* ad *B*, & vi sola *N*, ab *A* ad *C*, compleatur parallelogrammum *ABDC*, & vi utraq; feretur ad eodem tempore ab *A* ad *D*. Nam quoniam vis *N* agit secundum lineam *AC* ipsi *BD* parallelam, hæc vis nihil mutabit velocitatem accedendi ad lineam illam *BD* a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam *BD* sive vis *N* imprimatur, sive non, atq; adeo in fine illius temporis reperietur alicubi in linea illa.

Figure 2

by, and in a sense based them on, observation and experiment; (3) he (as do his scientific descendants to this day) predicted new phenomena and the results of untried experiments by the use of his general "laws."

Thus *The Principia* included the derivation of the orbits of the planets and the moon and studies of the shape of the earth and of the movements of the tides. All were derived as a consequence of his three axioms plus his "law of gravity": *all bodies attract each other with a force which is proportional to the product of their masses and inversely proportional to the square of the distance between them.*

Incidentally, there seems no real basis for the story that his conception of the law of gravity came as the result of being hit on the head by a falling apple. Sir D'Arcy Thompson's comment is most appropriate, however: "Newton did not shew the cause of the apple falling, but he shewed a similitude between the apple and the stars."

An interesting story about this so-called "inverse square law" gives added insight into both Newton's genius and some of his eccentricities. In 1684 Edmond Halley, the astronomer, Sir Christopher Wren, the architect, and Robert Hooke, a scientist, were talking together, probably in a London coffee house, about the motions of the planets, a popular topic for scientists of the day. Hooke said that he thought that obedience to an inverse square law would force the planets to move in the orbits which Johann Kepler had shown to be ellipses on the basis of the observations of his Danish master, Tycho Brahe. Sir Christopher Wren offered to buy Hooke a forty-shilling book if he would prove his assertion in two months. Hooke never claimed the book, but a few days after this Halley, in conversation with Newton, asked what orbit a planet would follow if it was attracted by the sun in accord with an inverse square law. Newton replied immediately "an ellipse." Surprised, Halley asked Newton how he knew, and Newton said, "I've calculated it."

Somewhat characteristically, Newton could not find the calculations at the time, but with much reluctance he was eventually persuaded by Halley to organize and write out his ideas in *The Principia*.

Halley then saw the book through the press and even paid for it upon its final publication in 1687.

Newton's basic ideas for his mechanics and also for his work in optics and what was to become the calculus (he called it "fluxions") were all conceived in 1665–1666, at his country home. Here, by himself, at the age of 24, he had done a great deal of experimenting and thinking during a stay enforced by the closing of Cambridge University during a plague.

In 1676 Newton published his first paper in optics. It was criticized by Hooke, Linus, and Lucas to the extent that Newton was forced to devote much time to explaining and defending his views. He finally said, "If I get free of Mr. Lucas's business, I will resolutely bid adieu to it eternally, excepting what I do for my private satisfaction, or leave to come out after me."

There may be other reasons for the twenty-year delay in the publication of his work on mechanics and attraction.[2] But during this time we find a continuing pattern of *conflict* (with Hooke over priority with reference to Newton's first law and his inverse square law, with Leibniz with reference to the origins of the calculus, with Hadley over priority in the invention of the sextant) and of *reluctance or procrastination in publication*. (Fluxions begun in 1665 received their earliest publication by Newton in 1704 as an appendix on the quadrature of curves added to his book on optics. Others, Hayes in particular, had published explanations of Newton's fluxions earlier, but the first complete treatment written by Newton was published posthumously in an English translation by Colson.)

Note also in the quotation above Newton's motivation—"for my private satisfaction." Although he was motivated by an interest in the physical world, it was intellectual curiosity and not the drives of economic need or commercial invention which led him on. No doubt his development of fluxions was largely motivated by his need for a tool in the study of forces, velocities, and accelerations, but his work on cubic curves was mathematics for its own sake. His proof of the binomial theorem and his allied work with infinite series lie between fluxions and cubic curves in their immediate practical utility. And, of course, Newton, who never married, really had only occasional cause to worry about money matters. He inherited some income from his father, and in his latter years he held lucrative appointments as warden and master of the mint.

In optics, Newton's theory that white light was actually a mixture of colored lights, bent by different amounts as they pass from one transparent medium to a second, was based on experimentation with prisms. He then saw the implication of his generalization for the construction of refracting telescopes (for which he ground his own lenses). This led him to invent and construct a reflecting telescope in order to avoid the distortion due to fringes of color that he felt to be unavoidable in the use of lenses.

In this invention he was following up an earlier suggestion of James Gregory's, but Newton *saw from his theory* of light the advantages of a reflecting telescope and *made* one. Gregory had done neither.

Even in mechanics and astronomy Newton extended, generalized, and synthesized ideas which had occurred to Galileo, Kepler, Hooke, and others. He himself recognized this and is reported to have said that "If I have seen a little farther than others it is because I have stood on the shoulders of giants." In addition to the "giants" noted earlier we must mention René Descartes, whose work in algebra and analytic geometry was a necessary preliminary to Newton's, and even Isaac Barrow, who, though probably a lesser giant than Descartes, was Newton's teacher.

Barrow especially, but others, too, had

[2] See, for example, F. Cajori, "Newton's Twenty Years' Delay in Announcing the Law of Gravitation" in the History of Science Society book cited above.

come very close to inventing the calculus before Newton. Newton, however, saw general procedures and formulated rules where others had largely solved separate problems. Newton more than the others also saw the close relationship between the differential and the integral calculus. Here, as elsewhere, a most important facet of his genius was the perception of relationships and generalizations. These are essential to prediction, and thereby to the extension and application of any theory—to, in fact, the design of earth satellites and space ships.

Have you read?

GASKELL, R. E. "The Practice of Mathematics," *The American Mathematical Monthly*, October, 1957, pp. 557–66.

Students are always asking about the place of mathematics in the vocational world. Mr. Gaskell has given a different aspect to vocational mathematics. This is the "Practice of Mathematics," which is analogous to the practice of medicine. Today's world of business and industry calls upon the mathematician for many aids.

The author points out that in the practice of mathematics the professional seeks the mathematics to which the problem can best be fitted. Therefore, he must (1) recognize and analyze the problem situation, (2) formulate the problem situation, and (3) solve the problem stated in a mathematical manner; then (4) compute cases of interest to the client and (5) interpret the results to him. The author goes ahead to give some very interesting cases to illustrate what he means. He concludes that we need to develop more mathematicians whose thought is problem-centered as opposed to method-centered. This gives us a challenging idea.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

NORTON, MONTE S. "Developing Success Qualities in Our Future Scientists and Mathematicians," *School Science and Mathematics*, November, 1957, pp. 629–35.

Certainly we all are interested in developing successful people, but our problems are manifold in this endeavor. We need to know what qualities insure success, then how to teach so that these qualities will become a part of the student. What these qualities are and how they may be implemented is the purpose of this article. I can list only a few, such as, dependability through cooperative planning and delegating responsibility; goal-directed activity through evaluation of classroom activities; experimentation; logical thinking; creativity; self-expression; patience; modesty, and alertness.

This article will set you to thinking on how well you are now developing such qualities, and it will give much help for future work in this area. Your students will like to read it also.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

PICKARD, JOYCE. "A Girl Mathematician in Industry," *Pi Mu Epsilon Journal*, Spring, 1957, pp. 275–77.

What can a co-ed mathematics major do? I am sure you have had this question asked of you many times. This article gives one possible answer. This is an interesting story of a girl who applied her mathematical training to setting up codes for digital computers. It is interesting to note that she had opportunity to travel, opportunity for advancement similar to all other employees, and that computing machines are charmed by women.

She also explains simply some of the steps a mathematician takes in getting a problem ready for the machine. You will want your girls to read this.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

ROWAN, HELEN. "The Wonderful World of Why," *Saturday Review of Literature*, November 2, 1957.

"The Wonderful World of Why where children are expected to think like children," is the way the author starts her article on the new approach to algebra. This is a view of the mathematics program at the University of Illinois as seen through the eyes of, as the author says, a "Mathematical Moron." She has an interesting style and presents the program in a way the non-mathematician can understand. Your students will be interested in reading the article; you will be interested in the editorial following, called "Explorations Unlimited." This editorial points out the need for modern mathematics and the progress that has been made only recently. It reveals how the study of mathematics will be a study of structure rather than manipulation and how this progress is all to the good. Mathematics is dynamic.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Some family finance topics for seventh and eighth grade mathematics

by Margaret Lingo, Pierce Junior High School, Grosse Pointe, Michigan

The teaching of practical, basic family-money management is one of the most challenging spheres of education. Students need to understand about financial planning, consumer credit, savings and Social Security, operation of banks, life and property insurance, stocks and bonds, housing, wills and estates—essential preparation for day-to-day living.

There is no one correct way to do this, but some basic objectives that everyone can follow and adjust to fit his situation can be set up. Bigelow[1] takes the view that a major problem facing the American family is to adapt itself to changing conditions both within and without the home in ways that will make available the richest possible life for all its members through the years. He feels that the quality of satisfaction is just as important as the quantity of satisfaction enjoyed by the family.

Education for wise use of family finances is not something that can be reduced to five or even forty lessons. To be successful it must be a continuing, planned attempt to coordinate and integrate the diverse fields of thought that have something to contribute to the shaping of an individual's standards of choice. It must also help him apply what he knows to his own particular situation when and wherever it fits his own needs and wants and at his own level of maturity and understanding. Teachers of junior high school mathematics can use topics from *Education for Family Finance* to enrich and implement their present programs.

The wise use of consumer credit might be selected as one important phase of the establishment of family financial security, since most young people in school today will use consumer credit in their adult lives. About sixty-five per cent of the major durable products sold today reach the family by some credit route. According to the "Survey of Consumer Finances," made in early 1956 by the Survey Research Center of the University of Michigan and published in the *Federal Reserve Bulletin*, forty-five per cent of the fifty-five million spending units had some short-term consumer installment debt. The typical debtor can be described as the head of a family, between twenty-five and thirty-four years of age, with one or more children below the age of eighteen years, and with an income of $3,000 to $7,500 per year. Of this typical group four out of every five families had short-term installment debt.

The consumer credit that young people will use will be their own. They should know when and how to use it to their advantage. The result could prove advantangeous to the nation's private enterprise

[1] Bigelow, H. F. *Education for Family Finance* (Philadelphia: J. B. Lippincott Co., 1953).

system, and the final result can and should be higher standards of living, better and more lucrative employment for labor, and prosperity in the national economy. Consumer credit is an economic tool that can be used effectively if people understand what it is, what they can expect of it in use, and how to handle it with skill and intelligence.

In developing a unit with a group of students the teacher will probably want to establish, either alone or with the class, goals in the form of concepts, skills, and attitudes which will be the desired outcomes of the planned educational experience. Some of the learning products, which are interrelated and interdependent, expected from a unit on Consumer Credit are:

Skills

1. Ability to compute finance charges on installment buying
2. Skill in computing interest and rate of interest
3. Ability to compute the true rate of interest
4. Ability to select the credit plan best suited to the family's needs
5. Ability to read the contract before it is signed
6. Ability to establish a satisfactory credit rating
7. Ability to manage credit wisely

Concepts

1. Understanding of the place of consumer credit in the family's financial plan
2. Recognition of the costs of consumer credit buying
3. Understanding of the advantages and the disadvantages of consumer credit buying
4. Understanding of the difference in cost between a credit purchase and the cash price
5. Realization of the importance of establishing intelligent buying habits
6. Understanding that consumer credit has a place in modern American culture
7. Realization that good habits of consumer buying can raise the standard of living of the family

Attitudes

1. Appreciation of the fact that consumer credit can be used wisely by a family
2. Willingness to pay credit charges for deficit financing
3. Appreciation of a good credit rating for the family
4. Appreciation of the state laws that regulate the handling of credit through lending agencies
5. Appreciation of the place of consumer finance companies in our economy

A group of Key Questions, such as the following, might be considered in class discussion and might be the basis for extensive reading and research.

Key Questions

1. Where can we go to borrow money? Which institutions are best?
2. What is meant by "true interest"?
3. When should we borrow money to pay for durables instead of buying on the installment plan?
4. How has credit helped expand our economy to the highest standard of living ever enjoyed by a nation?
5. Why is it important for a person to have a good credit rating?
6. What are the advantages and the disadvantages of installment buying?
7. How does a person finance an automobile?

Supplementary materials are available in the form of films, filmstrips, and pamphlets; and the local bankers and loan agents are usually quite willing to address the class and answer questions.

Evaluation is a continuing process and should be a part of the entire learning experience. Personal growth and progress should be reflected through understandings and changed attitudes and behavioral patterns and should be measured in terms of the objectives set forth. Much evalua-

tion can be of an informal nature, pointed out by the questions asked and the participation in discussion in group work. However, a written test can be administered as a pre-test and again as a final check to determine growth in each area considered. Many topics from *Education for Family Finance* have merit for the junior high school student if the teacher will remember to relate the material to the student's wants and needs and to his position in the family life-cycle.

Coefficients in expansions of squares of special polynomials

If we develop 1^2, $(1+x^n)^2$, $(1+x^n+x^{2n})^2$, $\cdots$ $(1+x^n+x^{2n}+x^{3n}+\cdots+x^{(m-1)n})^2$, where $n=1, 2, 3, \cdots$, and arrange the coefficients in form of a triangle, we obtain

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 2 & 1 & & & & \\
 & & & 1 & 2 & 3 & 2 & 1 & & & \\
 & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
1, & 2, & 3, & \cdots & m-1, & m, & m-1 & \cdots & 3, & 2, & 1
\end{array}$$

This enables one to write down the square of any polynomial of the above form, without performing the operating of squaring.

Example:

$$(1+x^2+x^4+x^6+x^8)^2=1+2x^2+3x^4+4x^6+5x^8+4x^{10}+3x^{12}+2x^{14}+x^{16}.$$

When the signs of the terms of the polynomial are alternating, then the signs of the coefficients in the expansion are also alternating.—*D. Mazkewitsch, University of Cincinnati, Cincinnati, Ohio.*

Coefficients in expansions of powers of special trinomials

Consider $(a+bx^n+ax^{2n})^m$, where $m=0, 1, 2, \cdots$, $n=1, 2, 3, \cdots$, and a and b any numbers, rational or irrational, real or imaginary.

Take any fixed values of a and b and expand the trinomial for $m=0, 1, 2, 3, \cdots$. Arrange the coefficients in form of a triangle. Then each number is the sum of the number in the row above it multiplied by b and the sum of the two adjacent numbers in the same row multiplied by a.

For instance, $(2+3x+2x^2)^m$:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 2 & 3 & 2 & & & \\
 & & 4 & 12 & 17 & 12 & 4 & & \\
 & 8 & 36 & 78 & 99 & 78 & 36 & 8 & \\
16 & 96 & 280 & 504 & 609 & 504 & 280 & 96 & 16
\end{array}$$

Here is, for instance, $504=78.3+(99+36).2$, $99=17.3+(12+12).2$, etc.

Example:

$$(2+3x^2)^3=8+36x+78x^2+99x^3+78x^4+36x^5+8x^6.$$

When the signs of the terms of the trinomial are alternating, then the signs of the coefficients in the expansion are also alternating.

It is interesting to note that when $a=1$ and $b=2$, then the coefficients in the expansion are the binomial coefficients in the expansion $(x+y)^{2m}$.

Example:

$(1-2x^3+x^6)^4=1-8x^3+28x^6-56x^9+70x^{12}-56x^{15}+28x^{18}-8x^{21}+x^{24}$.—*D. Mazkewitsch, University of Cincinnati, Cincinnati, Ohio.*

A column of unofficial comment

Breaking through the know-how barrier

by H. Van Engen, University of Wisconsin, Madison, Wisconsin

Recent events have emphasized the fact that more than ever before in the history of civilization mathematics teachers are "in the front line" of scientific efforts of this country. America needs more and even better scientists if for an extended period of time we are to keep the world stable enough to enable us to make a peaceful settlement of our differences with other nations. In the meantime you and I, as teachers of mathematics, have a job to do.

In the days past, teachers often felt that having an A.B., M.A., or Ph.D. was an automatic license to let up on their study of mathematics. Of course this never was justifiable. (A teacher engaged in teaching others to learn mathematics should be actively engaged in that project himself.) However, today it is even less justifiable. Mathematics has advanced and changed its point of view during the past few decades. This forward movement of mathematics has produced a phenomenon which is probably unique in the history of mathematical education. Research in the foundation of mathematics, mathematical logic, and mathematics itself have produced some very profound implications for a change in content for grades 1 to 14. We as teachers are already aware of some of these demands for a change in our subject's content. And only through further study can we become fully aware of why these changes are being requested.

However, there is another aspect of our present-day situation which needs much thought. A change in content is not enough. There is a vital element in all mathematical instruction which must be given considerable emphasis if we as mathematics teachers are to do our share in producing future scientists.

Too frequently, we mathematics teachers lose ourselves in the "machine" aspect of our subject. At times, we drill and drill the pupil in hope that this will "teach quadratic equations," thereby forgetting that no mathematician is a machine, and that industry builds machines to solve equations. Mathematicians continually strive for abstractions and generalizations. If we are to produce good mathematicians, then our instructional procedures must be so oriented that students are asked to abstract and generalize. Our budding mathematicians must develop the habit of abstracting and generalizing, and teachers must help them develop this habit.

It is in the abstracting and generalizing area that we need a major break-through. Change the content as you will, if the pattern of instruction doesn't change with the change in content, then we will debase modern mathematics (and future mathematicians) just as we have debased the mathematics that is now in our curriculum. Modern mathematics is a point of view; a habit of looking for the general; a willingness to pull out the essence. Omit these from your objectives and we can teach sets, linear programming, matrices, etc., but we will not be teaching mathematics *and* we will not be doing what we should to produce more and even better mathematicians.

Learning to teach for abstractions and generalizations is not easy. The teacher must "sway" with the class; he must depart from the text at a moment's notice; he must seize upon the smallest evidence that a student is struggling with an important generalization; he must know how to set the stage for abstracting and generalizing just as a director of a play must know how to manage the cast of a play, and the stage itself, so as to communicate the play's message to the audience.

When more teachers break through this know-how barrier, and when we replace some of the mathematics now being taught in the schools and colleges, as well as change the spirit of that now in existence, then it will be possible for us as teachers of mathematics to really serve in the "front lines" of our national effort.

What is an American college for, anyway?

A big part of the future job is to see that only those who will profit from the college experience be encouraged to enter. There is being propagated an alarming idea that virtually every high school graduate, regardless of motivation and aptitude, should have the privilege of attending a college of some sort. If this propaganda succeeds, the term "higher education" will be as perverted in America as is the term of "people's democracy" in Russia. Young people lacking the qualities which enable them to succeed in college should go to work. What has happened in the secondary schools since the upward extension of the school-leaving age should constitute a lesson to collegiate educators.

The high schools have been mandated to accommodate pupils without much ability to learn from books, which, after all, are an unavoidable tool of education at all levels. Our national zeal for education plus early shameful exploitation of child labor has resulted recently in holding too long in secondary schools many boys and girls who should have gone to work. . . . —*President John A. Perkins, University of Delaware.*

Percentage of math and science students swings upward

Last year, for the first time since 1910, the percentage of high school students enrolled in courses in science and mathematics increased over that for the previous year, according to a recent study made by the U. S. Office of Education on enrollment trends in science and mathematics.

Even though the number of students enrolled in science and mathematics has steadily increased since 1910 along with the total high school enrollment, the percentage of those studying science and mathematics has gradually declined. For example, in algebra alone, although the total number of students enrolled in this subject increased from about 500,000 in 1910 to more than 2,000,000 in 1956, the percentage of high school pupils taking algebra decreased from nearly 56.9% in 1910 to a low of 24.6% in 1952, then increased to 28.7% in 1956.

The study shows that the percentage of public high schools offering courses in chemistry or physics at the 12th-grade level increased from 77% in 1954 to 82% in 1956. Geometry courses offered in the 10th grade increased from 78% to 81% in the same period.

The complete study on enrollment trends in science and mathematics will be published later this fall and will be available from the U. S. Government Printing Office, Washington 25, D. C.

—*Taken from* Ed Press News Letter, *October 16, 1957.*

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Elements of Mathematics (second edition), Helen Murray Roberts and Doris Skillman Stockton (Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1956). Cloth, x+308 pp., $3.50.

Our colleges are accepting a number of students who have had no secondary mathematics or whose retention is so poor that they cannot succeed in intermediate algebra. This text has been written particularly for such a group, and gives quite detailed explanations of elementary matters beginning with arithmetical topics. Since several intermediate algebra topics such as logarithms and progressions are omitted, there are possibly few high school courses that would be based on this selection of content. However, junior colleges and colleges that at present offer intermediate algebra followed by college algebra might well consider this text as alternative to the intermediate algebra, in order to provide a more adequate introduction and to avoid duplication with college algebra.

Students with little or no introductory algebra are likely also to have omitted geometry and should profit from the two chapters on mensuration facts of geometry.

A chapter on decimal fractions includes a unit on significant figures and computation with approximate numbers. Such topics are not uncommon in texts at this level but the authors here actually make considerable use of their conventions in measurement problems in later chapters, and this is an innovation!

The exercises appear adequate in variety and quantity. The excellent practice is followed of providing answers to half of the problems and making the others available to the instructor. This will be particularly appreciated by college instructors, many of whom consider secondary texts unusable if they omit all answers.

The format and type are pleasing, and the book makes an excellent over-all impression. However, the reviewer was disappointed with regard to the claim in the preface that there is offered a "logical presentation of the number system." In his opinion there has not been much advance here over other texts. That hardy perennial of meaningless expressions, the literal number, is still retained. He also doubts if there is any gain, pedagogical or mathematical, by avoidance of the names Associative, Commutative, and Distributive, and the use instead of Laws of Grouping, Order, and Common Factor, respectively.

Addition is defined as the operation of finding, *without counting*, a single number which is equal in value to the combined value of two or more numbers. Why insert "without counting"? Is a criterion for efficiency a proper part of the definition? This "definition" appears widely. Would its users admit that an adding machine can add?

The absolute value of a number is said to be its value without regard to sign. Students have a tendency to infer from such a statement that the absolute value of $-a$ is a.

There are other statements that do not meet the reviewer's criteria for a logical development. To be too critical of these, however, would leave a negative impression of a book that definitely merits consideration in its field.—*L. Clark Lay, Pasadena City College, Pasadena, California.*

How to Solve It (second edition), G. Polya (New York: Doubleday and Company, 1957). Paper, xxi+253 pp., $0.95.

Professor Polya attempts in this book to make explicit some of the general principles and procedures which apply to the behavior that we call problem-solving. The types of problems used as examples are mathematical and fall into the categories of "problems to find" and "problems to prove."

Part I of the book deals with a list of questions which one may ask himself or which a teacher may ask a student. These questions are truly leading questions and are products of the principles of problem-solving advanced by the author. The questions are grouped into four sets—one set for each of the four stages of problem-solving. These are: 1) Understanding the Problem, 2) Devising a Plan, 3) Carrying out the Plan, and 4) Looking Back. Examples of the questions are: "What is the unknown? What are the data? What is the condition?" "Do you know a related problem?" and "Can you use the result [of the problem solved], or the method, for some other problem?" (pp. xvi and xvii).

Part II is a very short dialogue illustrating the use of the author's questions. Part III is called a "Short Dictionary of Heuristic," but it is arranged more like an encyclopedia. Part IV includes twenty problems along with hints

for their solutions and the solutions themselves.

In the opinion of the reviewer, Professor Polya should be commended highly for his work. Teachers of mathematics should consider very carefully what he has written in this book. Learning how to solve problems is acknowledged by many as the primary objective of instruction in mathematics at the elementary and secondary levels. It follows then that we teachers of mathematics ought to study any proposed theory concerning the mental operations involved in problem solving.

Polya emphasizes the view that "Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science" (p. 177). This book is more concerned with the latter than the former. The reviewer feels that mathematics ought to be "in the making" for each student when he discovers and learns something in mathematics.

The treatments of *analogy, auxiliary elements, generalization,* and *induction and mathematical induction* found in Part III are exceptionally good. Each is explained and illustrated by examples.

The only weakness detected by the reviewer is that of the organization of the book. The separate parts do not build upon the preceding parts in an essential way. The "dictionary of heuristic" in Part III causes an undue amount of repetition. These weaknesses are minor, though, when compared with the strengths of the book.

The reviewer would class this book as a "must read" book for every secondary teacher of mathematics.—*Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa.*

Refresher Arithmetic, Edwin I. Stein (Boston: Allyn and Bacon, Inc., 1957). Cloth, xiv +434 pp., $3.36.

Here is an abundance of items for practice. The book provides a wealth of reference material, a host of good exercises, and a noteworthy supply of interesting problems.

This arithmetic, moreover, is well organized; it progresses gradually from the addition of whole numbers through the various operations with fractions, decimal fractions, and per cents. It treats intuitive geometry, measurement, and formulas. Inventory tests, diagnostic tests, and achievement tests appear in strategic places. Accompanying the tests are references to exercises designed to correct pupils' computational deficiencies. Pupils can concentrate on specific materials, and work toward their own self-improvement.

Physically, too, the book is ample. It is well put together. Its art work is attractive.

Clearly the book fits pupils in their teens. Despite a statement, in the preface, that "it is designed for use in arithmetic classes at any grade level," the book's references, examples, and problems appeal mostly to adolescents. And, whether we embrace or deplore the fact, people who need a refresher in arithmetic do tend to be young adults.

Presumably pupils forget, even though they may have heard excellent explanations, seen excellent examples, and practiced excellent exercises. For those who believe that a refresher arithmetic will reinforce the bonds of learning, Mr. Stein's book fills the bill. To those who believe that once explanations to be heard, examples to be imitated, and exercises to be repeated have missed their mark, then to require more of the same would be a questionable procedure—to such this book would have much less value.

Indeed, *Refresher Arithmetic* is just what its title suggests. It rests on tell-and-show pedagogy. Its approach stems from the assumption that careful explaining, appropriate illustrating, and meticulous practicing assure understanding. Experimenting, exploring, and discovering receive little, if any, encouragement in such a procedure. Only twice, on pages 229 and 359, are the pupils induced to make estimates. In those cases the pupils check their estimates by measurement. Estimating numerical outcomes prior to computing is not emphasized at all.

The book highlights the skills of arithmetic. Extracting square roots entails eleven steps. Long division involving divisors that are decimal fractions responds readily to carets and shifting decimal points. The latter get justification from the observation that the same exercises done as common fractions and via the caret method have the same quotients. Application of the principle that a number, such as 10.3/0.17, may have other names, such as 1030/17, is not used.

In brief, teachers who desire a good handbook and an excellent book of problems should consider using *Refresher Arithmetic.* But those prospective users who wish to emphasize the *why* of arithmetic over the *how* would need to supply such an emphasis themselves.—*I. H. Brune, Iowa State Teachers College, Cedar Falls, Iowa.*

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

Testing in college mathematics

by Sister M. Stephanie, Georgian Court College, Lakewood, New Jersey

All teachers of college mathematics test their students. They administer achievement tests when the student arrives at college to learn his mathematical maturity, something that is not always clearly represented by a certain number of entrance credits or a grade on his high school transcript. Thereafter, for four years, they test—bi-weekly, or monthly, or at the end of a unit of work, or at the end of a semester. They test in order to learn how much the student has assimilated, where the students' weaknesses lie, to see how effective their own teaching methods have been, and in order to provide additional motivation for study and review, for there is nothing like an announced quiz to make the student attain a fine overview of the subject!

College instructors have their favorite methods of testing; some, say the students, even have their favorite questions. Each man knows what he hopes to achieve by testing and plans the construction of his examination accordingly; and this is as it should be. However, the purpose of this paper is to make a plea for standardized tests in college mathematics such as do not exist now, or exist only for certain subjects. The content of the college mathematics course, whether it be given in a liberal arts college or in a teachers' college, is changing, as is well known. New trends are every day becoming more apparent: "modern" mathematics, formerly reserved for the graduate school, is now appearing in the freshman course; subjects that were formerly considered entirely separate are now being successfully fused; and great emphasis is being placed upon meaning rather than upon skill in manipulation. In spite of all this there is a certain constant body of knowledge, traditional though it may be, which still constitutes, and will for some time continue to constitute the core of the college mathematics curriculum. It is for this common material that it would seem tests would be useful.

For years the Division of Chemical Education of the American Chemical Society has been constructing and marketing tests on all the phases of chemistry: general, organic, analytical, physical, general with qualitative analysis, general without qualitative analysis, etc. These tests exist in several forms, are revised periodically, and have been administered to so many students in colleges throughout the country that reliable norms are available. The tests are prepared by a group of college chemistry teachers (volunteer), who furnish items on a particular topic. A mimeographed copy of a particular test is then sent to all those who furnished items, but this test is about three times the length of the final test. Items are examined for content, difficulty, and correctness of expression and graded accordingly. Then members of the group meet to discuss each item on the test in the light of the criticisms given. Finally, the test in its completed form is sent to a sampling of colleges, and

tentative norms are established a year before the test actually appears for publication. All of this represents a tremendous amount of work, particularly on the part of those directly in charge of the construction of each test, but the tests are a highly respected method of measuring the progress of an individual chemistry class against the national average.

It is also possible to obtain the *Cooperative Physics Tests for College Students* in mechanics, heat, light, electricity, sound, and modern physics. These tests, too, are available with norms, and in a sufficient number of different but equal forms so that the test need not be repeated from year to year. In addition the American Council on Education publishes such tests as "Fluid mechanics—college level," and "Electricity and Magnetism—college level."

Educational Testing Service last year published a large loose-leaf book of multiple-choice questions in all branches of science which can be used as a reservoir from which to draw test items for almost any college science course.[1] There are no mathematics questions except those which directly pertain to some science such as chemistry or physics.

Surely the material of college mathematics lends itself to testing as well as that of college chemistry and physics. Could there not be made available through some already existing association or agency, sets of mathematics examinations at the college level in such subjects as the differential calculus, or analytic geometry, or a combination of these? One can now purchase tests in solid geometry and in trigonometry, but these can hardly be classified as college subjects. I know of many, many college algebra textbooks, for example, but of no test if one exists.

I am thinking of something on the same level and with the same type of question as the College Entrance Examination Board's *Advanced Placement Test* in mathematics. This, according to an article in THE MATHEMATICS TEACHER, October, 1957, covers the standard freshman course in analytic geometry and calculus.[2] The purpose of this test is to assist the individual college mathematics department in the placement of students who have had advanced mathematics in high school, and in the granting (or not) of college credit for such subjects. The items are not pretested but "the complexion of the Examiners Committee is such that one could expect only an eminently sound examination." From the sample questions given in this article, and from those appearing in the *Advanced Placement Bulletin*, one can see the excellent quality of the test.

College mathematics tests have been prepared for Educational Testing Service by a committee of noteworthy mathematicians for the advanced mathematics section of the Graduate Record Examination. Why cannot the same kind of examination be made available, in the usual way of ETS examinations, but compartmentalized by subject, or combination of subjects? For a beginning, only those subjects which are taught in every undergraduate curriculum could be prepared, and then later, subjects which appear more infrequently. Even if more and more modern mathematics is taught, the differential calculus and integral calculus will be part of a college course for some time to come. The Graduate Record Examination seems to be based upon the premise that there is certain material of which every college graduate with a major in mathematics should be aware, and that there is other material which is optional—no one in four years is expected to have had all those courses. This paper is concerned with the knowledge that every one should have, and the testing of that knowledge.

[1] Dressel, Paul L., and Nelson, Clarence H., *Questions and Problems in Science*, Test Item Folio No. 1 (Princeton: Cooperative Test Division, Educational Testing Service, 1956).

[2] Douglas, Edwin C., "The College Entrance Examination Board's Examination for Advanced Placement in Mathematics," THE MATHEMATICS TEACHER, L (October, 1957), 458–461.

One might raise the objection that mathematics teachers all have their favorite ways of teaching, no two teachers organizing material in exactly the same way, and so they will prefer to construct their own examinations. This is surely true for tests given during the progress of a course; but there must be many instructors, particularly those who have not been teaching very long, who would use a standardized test at the end of a semester to measure the progress of their class. The material of college mathematics, especially that of the first two years, is sufficiently alike in colleges throughout the country that the same basic information is imparted to all. Even if, for example, there are ten sections of differential calculus, the ten instructors are not always left free to construct their own individual tests. They frequently work together to construct one test that is administered to all the students. If this is done, one assumes that in that college the material taught is the same for all.

One might also say that an objective test of the kind suggested is open to guessing, or does not adequately test reasoning powers or the ability to work through, in all its steps, a complete problem. Then let there be, if necessary, a section on each test, as there is on the Advanced Placement test, where detailed answers are required. Introduction of this element of subjectivity makes, of course, the tests more difficult to correct and to grade accurately. It seems that a well-constructed objective test, with a sufficient number of choices in each item to obviate guessing, would serve the same purpose just as well. The Graduate Record Examination is accepted throughout the country, its scores are considered in the awarding of graduate fellowships and scholarships, and there is not an essay question on it.

The preparation of such suggested examinations would mean a tremendous amount of work in formulating items, pretesting, and so on. Perhaps such a plan has already been made and such tests are in the making. Or again, perhaps there exist good reasons why the plan is not at all feasible. There seems to be very little periodical literature of any kind on the evaluation of college mathematics instruction, although there is much on high school testing. It cannot be that all college instructors are completely satisfied with their present methods, and even if they were, one must consider all the future college instructors who will be neophytes during the next few years. If the rising enrollments of colleges continue, there will be many of these, and they will be the ones charged with the teaching of the first few college years of mathematics. There is great need of a method whereby, at the end of a year, they can measure what their class has accomplished, in comparison with other classes in the same subject throughout the country.

"It is impossible not to feel stirred at the thought of the emotions of men at certain historic moments of adventure and discovery—Columbus when he first saw the Western shore, Pizarro when he stared at the Pacific Ocean, Franklin when the electric spark came from the string of his kite, Galileo when he first turned his telescope to the heavens. Such moments are also granted to students in the abstract regions of thought, and high among them must be placed the morning when Descartes lay in bed and invented the method of co-ordinate geometry."—*Alfred North Whitehead.*

Plans for the Journal[1]

by W. W. Sawyer, University of Illinois, Urbana, Illinois

The situation of American mathematics is a great paradox. The general attitude of Americans to life is intelligent, enterprising, non-traditional. In mathematics teaching the opposite obtains; everything is traditional, based on rote learning, the acceptance of authority, the absence of understanding. It is only the exceptional efficiency of Americans in other spheres of life that has enabled America to survive with a mathematical syllabus in which the average child is two years behind his European counterpart and the able child at least four years. And the European system itself is far from being a model of how mathematics should be taught. Indeed in Europe, as here, the efficiency of mathematical teaching is not merely low, it is a negative quantity—it puts pupils against mathematics in a very large number of cases.

Sooner or later, as mathematics became increasingly necessary for technology, a crisis was bound to come. Sputnik has only dramatized the situation already existing.

Anything less than radical change will fail to meet the situation. No program should be criticized because it implies a complete overhaul of syllabuses, the writing of new textbooks, and a changed training of teachers. Rather, any program without these things is inadequate.

Most difficulties with mathematics are emotional rather than intellectual; fear and anxiety, boredom, a feeling that the subject is useless hairsplitting, or even when useful is dull—these are the causes that drive pupils from mathematics.

Now, our tastes are formed early in life. Early impressions are lasting ones. If at ten years of age a child hates mathematics, that attitude will probably be lifelong.

In grades one and two, most children love arithmetic. The trouble comes later. Grades five through eight are probably the worst spot. They form a vacuum in which practically nothing is chewed over again and again. I have found that *both the ablest and the least able* pupils in these grades respond enthusiastically to a mathematically richer syllabus. I hasten to add that I do not mean anticipating the formal, verbose, rote-learning approach to algebra by four years, but giving the pupils the experience of making mathematical and scientific discoveries for themselves, in algebra and in physics.

Arithmetic as a subject of mainly *commercial* importance died with the coming of the desk calculator and the electronic computer. The significance of arithmetic today is as a source of pattern —the patterns of numbers leading to higher mathematics (via algebra) and to scientific laws (also via algebra). This viewpoint is far more intellectually stimulating; on this children and mathematicians agree.

The first step toward such a change would be to make enrichment material available from grade five onward. This should be material designed to be read by the children themselves. Eventually, when we had discovered what produced the greatest enthusiasm in children and how much they could absorb, and when

[1] W. W. Sawyer has been appointed editor of *The Mathematics Student Journal*. His first issue will appear late in 1958.

teachers had become accustomed to the new material, the fresh topics could gradually be absorbed into the standard texts.

The abler children constitute no problem at all. They should be encouraged to read on for themselves, which they are usually glad to do. A list of books should be prepared and circulated to all schools: "if you like mathematics, read these books in, more or less, this order."

No one can object to this on democratic grounds, since it would absorb no teaching manpower at all. By the age of thirteen I should expect a mathematically-gifted child to have covered a good deal of algebra, a very rapid and condensed survey of Euclid's main geometrical theorems, some trigonometry, the beginnings of analytical geometry. Calculus could then come early in high school work. Less gifted pupils would do some but not all of this. An essential point is that we should immediately discard the idea of a class advancing at a uniform pace under the constant supervision of a teacher.

Euclid is no longer regarded as a model of logical thinking. To make geometry a logical subject is a matter for graduate school. But a student should know the results of Euclid, and see that some kind of plausible reasoning relates certain theorems. Altogether there are about thirty theorems in Euclid worth knowing. The whole of these can be exhibited on one large sheet of paper. These results are useful for trigonometry and for analytical geometry.

I envisage abler students learning nearly all the valuable part of the present high school syllabus before they enter high school. This is the only satisfactory solution to the argument between the traditionalist and the radical mathematicians. Modern mathematics grew out of traditional mathematics and is unintelligible without it. Further, the national crisis is in *applied mathematics*. Relatively few parts of modern mathematics have yet found scientific uses. Of course, they may do so in future. But the applications of modern mathematics usually require traditional mathematics as well. Measure theory makes a contribution to statistics—but you need calculus to do statistics; topology makes a contribution to airplane design—but via the traditional subject of differential equations. It is noteworthy that the opening chapter of Lefschetz' *Introduction to Topology* deals largely with topics using calculus.

In short, a mathematician needs familiarity with a wealth of topics from traditional, concrete mathematics, which he then refines, classifies, and unifies by the methods of modern mathematics. The only way to get this is to start young—there is so much to learn. The last two years of high school could be occupied with such subjects as matrices and with topics from modern mathematics. Matrices form a kind of bridge between ancient and modern. They are concrete; you can calculate with them; they have important applications in modern science and technology; and they can be used to illustrate some modern mathematical concepts.

"How" is as important as "what." Unless pupils continually increase their enthusiasm for mathematics, they would be better off not learning it at all. *The Student Journal* sets itself the task of finding a presentation of mathematics that will stimulate pupils to advance and to advance happily.

In view of the early stage at which changes are needed, *The Student Journal* intends to widen its scope and to appeal to all grades from seven through twelve. If some can use it in grades five and six, good luck to them.

Obviously it will only set pupils back if *The Student Journal* assumes a higher standard than that which exists today. Its contents must have the form of an inclined plane—beginning with articles and problems that assume nothing outside arithmetic, and extending to the hardest questions considered by the seniors of the most mathematically-enterprising schools. The aim is to demonstrate to the youth of

America that they can do these things earlier than people think they can.

In particular, we should like to see textbook material covering the whole range of mathematics, by which the student can discover and develop the subject for himself. We should welcome articles showing a pupil of any age how to do this. We should be most interested to hear of any result actually discovered by a pupil or a group of pupils; of course, the result can be one that is already known to mathematicians.

When pupils start investigating, they often propose very difficult problems—sometimes problems to which modern research has not found the answer. We should like to hear of any question raised by pupils to which the teacher does not know the answer. I hope no teacher will feel ashamed of not knowing the answer; there is so much mathematics today that no one knows it all; research workers continually have to seek help from each other.

It is difficult for a new editor to guess how far articles and problems are meeting the needs of pupils in the thousands of schools in this country. It would be very much appreciated if teachers or pupils would report what they find too difficult or too easy, what interesting and what dull. Teachers' societies could help by appointing a particular individual to convey criticisms, requests for articles, etc. We should like to have at least one official correspondent in each state.

Could I also suggest that teachers who are interested in our approach might file some copies of *The Journal*? *The Journal* is a small paper. We cannot repeat in 1960 what we said in 1959, even if a new generation of pupils has entered the classroom. We hope in the three years 1958–1961 to make some contribution to the necessary changes in education; if you can file a few spare copies, your classes of 1961 may well find that the whole series makes some kind of sense—at least, we hope so.

Have you read?

SEIDLIN, JOSEPH. "Between the Devil and the Deep Blue Sea," *Mathematics Magazine*, May–June, pp. 277–281.

"Johnnys Multiply All Right, But They Sure Can't Add." This might be the title of a book calling attention to the attacks on mathematics that have been going on for centuries. At the elementary and secondary school level it appears the teachers use a vehicle, be it mathematics or history, to develop the student to his maximum. Since mathematics is not the only vehicle we cannot take all the responsibility, but what can we do to carry our share? First, understanding is only a part of learning. Learning must be continuous and consistent. Learning by students must be encouraged and, for the bright, accelerated. Learning is labor and a self-activity.

In relation to the above the author presents these general principles:

1. One who does not know his subject cannot teach.
2. Knowing is necessary but not sufficient.
3. A teacher is obligated to keep on learning.
4. A teacher is obligated to keep on learning the nature of the processes in learning.

"Mathematics is a Mountain. Vigor is Needed for its Ascent." This is a challenging article.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Thirty-sixth Annual Meeting

Hotel Cleveland, Cleveland, Ohio
April 10–12, 1958

Because of the times in which we are now living, this convention promises to be one of the most significant in our history. Arrangements have been made to provide for a large attendance, and the program provides for every level of instruction from kindergarten through college each day of the convention. The general nature of the program, described in the following paragraphs, will point to the value of representation at this convention from every school district of the nation.

High lights of the program are the general sessions and banquet. On Thursday evening, at 8:00 p.m., Professor Howard Raiffa of Harvard University will give us a down-to-earth introduction to the newest of mathematical developments, the theory of games. At the Friday morning session, Professor John G. Kemeny of Dartmouth College will speak on "The Discrete in Mathematics," or how simple counting becomes important in today's problems. At the banquet, Professor S. S. Cairns of the University of Illinois will provide an entertaining and instructive approach to conquering mathematical apathy. The Saturady morning session is honored with the author of *Mathematics in Western Culture*, Professor Morris Kline of New York University, who will speak on "What Mathematics Shall We Teach?" At the luncheon on Saturday, President Howard F. Fehr will give a brief but snappy talk on "Breakthroughs in Mathematical Thought."

Those interested in the elementary school will listen to such outstanding speakers as Herbert Spitzer, Glenadine Gibb, Jesse Osborn, Joe McKeeby Phillips, Catherine Stern, Nancy Rambush, John R. Clark, Foster Grossnickle, Marshall Stone, Newton Stokes, Robert Morton, Guy Buswell, and Allen Hickerson. Among the topics these people will treat are the meaning of symbols, the content and organization of arithmetic, meaning in arithmetic, ideas in a fog, the use of materials such as the Stern, Montessori, and Cuisenaire aids to learning, the mathematical structure of arithmetic, demonstration teaching by radio, and provisions for individual differences. This rich fare can not be missed by any teacher or supervisor or director of elementary arithmetic.

The junior high school sections provide topics on curriculum construction, the theory of arithmetic, building concepts, teaching problems, the use of models, modern mathematics in the junior high school, challenging content in the junior high school, and provisions for exceptional children. The speakers include Max Beberman, Max Sobel, Robert Fouch, Florence Elder, Henry Syer, Dan Dawson, John Reckzeh, Robert Swain, John Mayor, and Maurice Hartung. All these names have appeared in THE MATHEMATICS TEACHER from time to time and they guarantee a high type of challenging intellectual stimulation.

For the senior high school there will be talks and discussions on the nature and content of geometry and of algebra instruction, continuity of concept learning, strategy in concept formation, the unfolding curriculum in the high school, the use of models, the status and possibility of TV in teaching mathematics, illustration of modern mathematics and statistics for the secondary school, programs for the gifted, and provisions for the less able. The speakers include Frank Hawthorne, Lee E. Boyer, Professor Albert Tucker, Richard S. Pieters, Joseph Hooten, Henry Syer, Phillip Jones, Myron Rosskopf, E. H. C. Hildebrandt, Henry Van Engen, and Julius Hlavaty. Teachers will find in these high school sections much that they have been looking for in modern mathematics for the high school classroom.

For those interested in college mathematics and teacher training, there will be addresses by Houston T. Karnes, Jack D. Wilson, John E. Freund, Bailey Price, Franz Hohn, R. M. Thrall, Bruce Meserve, Karl H. Denbrow, Howard Levi, and Robert C. Yates. This group of outstanding college teachers will discuss the nature of geometry at the college level, modern geometry and modern algebra for teachers, application of mathematics in the social sciences, as well as application of relational algebra, college programs for freshman years and for non-specialists, and the over-all training of junior and senior high school teachers for teaching a modern high school program.

Among the special features there will be the showing of latest films in mathematics; the meeting of the delegate assembly on Thursday from 9:00 A.M. to 12:00 noon; a laboratory for elementary, junior high school, and senior high school teachers (with limited registration); a section on Cooperation with Industry; international programs in mathematics; and a Youth Forum with Professor H. J. Ettlinger as speaker.

For guests who come with delegates, as well as the delegates themselves, a number of trips and recreational activities have been planned that will add gayer moments to the convention. The complete program will be mailed to all members in February. Keep the dates—April 10–12, 1958—and reserve your room: Hotel Cleveland, Cleveland, Ohio.

I believe that the issue (*reasoning is mathematics*) should be resolved by equating formal logic with mathematics, as has been done here. It is of some importance for our later considerations that this equation recognizes mathematics as embracing much more than the study of quantity and number. Indeed, it requires us to conceive of mathematics as the study of general systems comprising designated objects and designated relations among them; and to regard the quantitative or numerical aspects of particular mathematical systems as accidental, rather than as essential or characteristic, phenomena for mathematics as a whole. It would, for instance, be wrong from this point of view to reject a scientific theory as nonmathematical merely because the theory is nonquantitative.—*Taken from "Mathematics and the Future of Science," by Marshall H. Stone,* Bulletin of the American Mathematical Society, *Volume 63, Number 2, March 1957.*

• NOTES FROM THE WASHINGTON OFFICE

New membership and subscription dues

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

"Because of the increased cost of printing . . ."—these were the words with which an announcement began in the April 1947 issue of THE MATHEMATICS TEACHER giving notice of an increase in dues from $2.00 to $3.00 per year.

During the eleven years that have followed this announcement notable changes have taken place in the Council. The number of individual members has increased about 170 per cent. The number of journals has grown from one to three. An active program of producing supplementary publications has been developed. Distinguished additions have been made to our list of yearbooks. This active program has required more travel by our officers and directors, increased activity by our committees, and an expansion of the clerical staff needed to administer the headquarters office. Meanwhile the cost of printing has continued to increase. Nearly all other costs such as those for clerical help, postage, supplies, and equipment have also increased.

For several years the increasing income from memberships, the active sale of publications, and the attempts of your officers to operate the Council efficiently were able to support the financial needs of the Council in spite of rising prices. We were able to increase our cash reserve each year until 1956. During the 1956–57 fiscal year, however, our expenses began to exceed our income, with the result that we experienced a deficit of about $8,000.

When the budget for the present fiscal year was adopted, it was apparent that our expenses again would exceed our income, perhaps to the extent of $15,000 or $20,000. Fortunately our cash reserve is still large enough to see us through one more year of deficit operation.

With some reluctance, but with strong faith in the future of the Council and in the loyalty of our members, the Board of Directors voted on August 18, 1957 to adopt the new schedule of membership dues and subscription fees shown below. The present arrangement of making a subscription to either THE MATHEMATICS TEACHER or THE ARITHMETIC TEACHER a part of the membership fee, with both journals available for a slightly higher charge, has been retained.

Individual membership	
Including either journal	$5.00
Including both journals	$8.00
Institutional subscription	
THE MATHEMATICS TEACHER	$7.00
THE ARITHMETIC TEACHER	$7.00

The student membership fee remains the same as now, with one journal available for $1.50 and both for $2.50. The subscription fees for *The Mathematics Student Journal* also remain unchanged.

These new fees will go into effect on October 1, 1958, and will apply to memberships and subscriptions as follows:

1. All orders for which the subscription year begins in October 1958 or later.
2. All new orders received October 1, 1958 or later, regardless of the beginning date.
3. All renewals of memberships and subscriptions that expire with the last issue of the present school year or a later issue.

Recent developments in science and engineering have focused attention perhaps more strongly than ever before in history on the basic importance of mathematics to the modern age. It is urgent in this critical time that the Council continue to receive the loyal support of its members and that its important program be not handicapped by a lack of financial resources.

Report of the Television Committee

by Joe Hooten, Chairman, Florida State University, Tallahassee, Florida.

Someone once referred to the television receiver as a "window on the world." Those of us who have watched our students come to class "the morning after," with a bleary-eyed look about them and a mind filled with visions of the likes of the Lone Ranger, have often wished that someone would pull down the shades.

Our wish was probably much more fervent a decade ago, though, when commercial television time was almost entirely consumed by old movies of the horse opera type. Of course there were, even then, a few brave souls to call television a potential tool for teaching and public relations work. But today their prophecy is actually beginning to see fruition. Some school systems have already instituted the closed circuit (or limited audience) network for instructional purposes. Indeed, some state systems are now in the televising business.

It is not unexpected when schools use television for educational purposes, however. What is really interesting is that this season there will appear on your screens two different series of programs on mathematics which will be produced and financed by the *commercial* networks.

The Westinghouse Broadcasting Company, Inc., has combined mathematics and marionettes in producing an educational film series called *Adventures in Number and Space*. These films feature such famous puppet characters as Snarky and are dedicated to "the novel proposition that TV can inspire children to look, learn, and like mathematics."

These films for televising were prepared by Bill Baird in co-operation with Teachers College, Columbia University. Dr. Howard F. Fehr, president of the National Council of Teachers of Mathematics, acted as consultant to Mr. Baird and collaborated in the writing of the scripts. There will be a total of nine half-hour programs, and the subject matter runs the gamut from "How Man Learned to Count" to "Stretching the Imagination." They cover such diverse topics as algebra, geometry, probability, statistics, topology, and careers in mathematics.

While the initial televising will be seen only from those stations belonging to the Westinghouse network, Westinghouse will subsequently make the films available to educational television stations and for classroom use nationally. They will be available shortly after the first of the year.

The National Broadcasting Company is producing a series of ten programs entitled *Mathematics*, with the central purpose of offering descriptions of the great breakthroughs in mathematical thinking, their impact at their respective time in history, and their influence today. The format for each program revolves about an interview between Clifton Fadiman, the host for each program, and an outstanding

guest-mathematician. The roster of visiting dignitaries includes such names as Robert Swain, S. H. Gould, Oystein Ore, Edgar Lorch, Frederick Mosteller, Howard Levi, R. L. Wilder, Paul Halmos, John Kemeny, and Brockaway McMillan.

The topics under discussion vary from the discovery of number to operational research and include such notions as symbolic logic, non-Euclidean geometries, and set theory. The programs will be televised "live" from the studios in New York and will be carried by some of the NBC stations throughout the nation. Those stations which do not carry the series "live" may air them at a later date via film. For further information as to when you can see these and the Westinghouse programs it is suggested that you call your local station.

So it seems that once again outside agencies may be pushing the teaching profession into doing something new. However, this may not be as true as it seems. From scattered reports there is evidence that many mathematics programs have been produced on local TV stations. Some of these were done by universities and colleges, some by local secondary schools, some by the mathematics teachers' organizations in their respective areas. Not too much is known about these efforts, since no compilation of individual efforts has so far been made.

For this year the Television Committee of the National Council of Teachers of Mathematics (of which the author is chairman) has been requested to make a study of the problems, procedures, and results in the use of television to teach or teach about mathematics. Already the Committee is receiving requests from teachers and groups for help in producing programs on television. So far, not much help *can* be offered.

The Committee needs your help in organizing their study. If you have been involved in the production of a television program about mathematics will you please send a description of your experiences to the chairman whose address is listed below? If you have seen any such programs and have opinions regarding their effectiveness, pro or con, tell us your reactions so that the Committee can begin to formulate a true picture of the present state of affairs.

Everyone is aware of the need for more mathematicians. From this need there grows the need for more students taking mathematics in high school and college. This gives rise to a need for better mathematics in those courses. Finally, all of this contributes to the great need for better understandings of the basic concepts of mathematics. To do the job required of us demands that our messages be dispersed to the maximum number possible and as quickly as possible. The television set may well be a "window on the world." How much of the world of mathematics will it reveal?

LEWIS SCHOLL
SILVIA VOPNI
PHILLIP JONES
DAVID WELLS
GEORGE ANDERSON
JOE HOOTEN, Chairman
Television Committee,
1230 Amsterdam Ave.,
New York 27, N.Y.

Annual business meeting

As required by the Bylaws of the Council, notice is hereby given that the Annual Business Meeting will be held in Cleveland, Ohio, on April 11, 1958.

Report of the Secondary School Curriculum Committee

by Frank B. Allen, Chairman, Lyons Township High School, La Grange, Illinois

The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics proposes to accomplish its mission by means of a four-year comprehensive and critical study of the curriculum and instruction in mathematics in secondary schools with relation to the needs of contemporary society. This proposal has been approved by the Board of Directors of the NCTM with the understanding that the study is to culminate in a report formulating proposals for the strengthening and improvement of mathematics education in the secondary schools of the nation.

The report would include recommendations for implementing these proposals and would provide direction for continuing efforts of the National Council of Teachers of Mathematics to improve mathematics instruction. The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics has direct responsibility for the study and the preparation of the report.

The need for a study and a report

A preliminary study by the Curriculum Committee of the Council on present practices of teaching mathematics showed uncertainty in, and dissatisfaction with, the present state of mathematics education. It indicated an urgent need to provide clarification and direction of mathematics education for mathematics teachers and administrators in the schools of the nation. In the past during periods of transition in educational outlook, studies which were national in scope have provided this kind of direction.

In 1916, a committee of the Mathematical Association of America, later supplemented by a grant from the General Education Board, began an extensive study of mathematical education, which culminated in 1923 in the report entitled "The Reorganization of Mathematics in Secondary Education." This report had a tremendous impact on mathematics education: it resulted in new curricula; it modified college entrance requirements; it introduced new programs in mathematics for students not preparing for college; it gave rise to new textbooks.

By 1935, American education was again in transition. A new committee was appointed to study the mathematics curriculum. This committee, a joint undertaking of the Mathematical Association of America and the National Council of Teachers of Mathematics, completed its work with the publication in 1940 of the Fifteenth Yearbook of the National Council, entitled *The Place of Mathematics in Secondary Education*. This report has had considerable influence and effect on mathematics education.

Now, over twenty years later, the inadequacy of those reports for present-day educational needs is evident. The Second World War and subsequent scientific and technical advances have changed our cultural pattern. Mathematics today has a new and important role in the industrial and social life of the nation. New and significant mathematics and applications of mathematics are appearing. These changes are continually emphasized by industry, government, colleges, scientific societies, and institutions concerned with mathematical knowledge.

Mathematics teachers need direction and assistance in bringing new views of mathematics into their classroom teaching. Most teachers of mathematics now in service are generally unacquainted with, and have had little or no opportunity to learn, certain contemporary concepts of traditional mathematics, some of the newer mathematics, and present-day applications of mathematics. The study and report proposed by the committee are designed to meet these needs.

The National Council of Teachers of Mathematics

It is appropriate that a committee of the National Council of Teachers of Mathematics undertake this study and issue this proposed report. The National Council is the largest and most comprehensive professional body whose primary concern is secondary-school instruction in mathematics. It can speak for classroom teachers, supervisors and administrators of mathematics programs, and mathematicians and educators responsible for the training of teachers of mathematics. It carries its message to these various professional workers through its three annual meetings and its several publications which circulate to more than 16,000 teachers and institutions. It has professional responsibility for the mathematical education of all youth—from kindergarten through the twelfth grade, from the least able to the highly gifted in learning ability—in both public and private schools.

The National Council is well aware that there are other organized groups, special in character, engaged in significant but limited studies of mathematics education. The contribution of these special groups will be recognized and utilized in this proposed study. However, there is no group other than the Council whose concern in this area is universal. Moreover, through its own membership and the assurance of cooperation by individuals and groups—mathematicians, scientists, educators, and industrialists—the Council is assured of high interest and exceptional ability in making the study and producing the proposed report.

For these reasons, a report with the greatest potentiality for improvement of the whole of secondary school mathematics can best be made, and properly should be made, by an agency of the National Council of Teachers of Mathematics.

Plan of action

The Committee will initiate a series of related studies to be prepared by subcommittees and will take cognizance of the work of related committees throughout the world. Subcommittees for study in each of the following areas are contemplated:

1. The place of mathematics in our changing society
2. The aims of mathematics education and the pedagogy of mathematics
3. The nature of mathematical thought for grades seven to twelve
4. How geometry should be introduced and developed
5. The content and organization of junior high school mathematics
6. Foreign mathematics programs for pupils of ages twelve to eighteen
7. Adjustment of the mathematics program to pupils of exceptional ability (scope includes both gifted and retarded)
8. Aids to teaching
9. The organization of the mathematics program
10. The administration of the mathematics program

All but one of the subcommittee chairmen have been designated, and each appointee has been provided with some carefully considered instructions, specifying the precise area to be considered and indicating the basic problems and issues on which recommendations are sought. (These instructions may form the basis for

a later report to be made after all subcommittee members have been appointed.) In forming subcommittees the following criteria will be employed:

There will be a member of the Curriculum Committee on each subcommittee;

Secondary schoolteachers will have adequate representation on each subcommittee;

Representatives of industry, research mathematicians, applied mathematicians, and individuals having experience in related studies will be included on appropriate subcommittees.

Although their methods of operation will vary, it is expected that each subcommittee will need several meetings plus secretarial assistance. Representation of the Curriculum Committee on each subcommittee insures coordination of the work of the various subcommittees. More than fifty persons of competence and established reputation will give of their time and counsel in the deliberations of the subcommittees identified.

After individual reports have been revised, amplified, and coordinated as may be necessary, the Committee will consider all subcommittee reports together with information collected from related groups and will devise a master outline based on this material which will describe the general organization and content of the final report. At the same time the Committee will formulate the editorial policy under which the final report will be prepared. At this stage (about April, 1960) the master outline will be turned over to a writing staff whose task will be to complete a preliminary draft of the final report as expeditiously as possible in order that all revisions deemed necessary by the Committee can be accomplished by October, 1961. Publication and distribution should be accomplished in early 1962.

MEMBERS OF THE SECONDARY SCHOOL CURRICULUM COMMITTEE OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Jackson P. Adkins, Phillips Exeter Academy, Exeter, New Hampshire. Teacher of Mathematics.

Kenneth Brown, Department of Health, Education and Welfare, Office of Education, Washington 25, D. C. Specialist for Mathematics.

Howard F. Fehr, Teachers College, Columbia University, New York 27, N. Y. Professor of Mathematics and Chairman, Department of Teaching of Mathematics.

A. S. Householder, Oak Ridge National Laboratory, Oak Ridge, Tennessee. Chief of Mathematics Panel.

Mrs. Lottchen L. Hunter, Wichita Public Schools, Wichita, Kansas. Supervisor of Mathematics.

Burton W. Jones, University of Colorado, Boulder, Colorado. Professor of Mathematics and Chairman of Dept. of Mathematics.

John R. Mayor, American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W., Washington 5, D. C. Director of Education.

Bruce Meserve, Montclair State Teachers' College, Montclair, New Jersey. Professor of Mathematics and Head of Dept. of Math.

Sheldon S. Myers, Educational Testing Service, 20 Nassau St., Princeton, New Jersey. Director, Mathematics Testing Section.

Mr. E. B. Newell, Allison Division, General Motors Corporation, Indianapolis 6, Indiana. General Manager.

Alfred L. Putnam, University of Chicago, 416 Eckert Hall, Chicago 37, Illinois. Chairman, Department of Undergraduate Mathematics.

Miss Elizabeth Roudebush, Seattle Public Schools, 815 Fourth Avenue North, Seattle 9, Washington. Director of Mathematics.

Helen M. Walker, Teachers College, Columbia University, New York 27, N. Y. Professor of Education–Emeritus.

Mrs. Marie S. Wilcox, Thomas Carr Howe School, Indianapolis 7, Indiana. Chairman, Department of Mathematics.

Professor Lynwood Wren, George Peabody College for Teachers, Nashville 4, Tennessee. Professor of Mathematics and Head of Department of Mathematics.

Frank B. Allen, Lyons Township High School and Junior College, La Grange, Illinois. Chairman, Dept. of Mathematics and Chairman, Secondary School Curriculum Committee.

Consultant:

Nagmus R. Hestenes, University of California, Los Angeles 24, California. Professor of Mathematics and Chairman, Department of Mathematics.

Summer institutes for secondary mathematics teachers

During the summer of 1958, the National Science Foundation is again sponsoring institutes for teachers of mathematics. Stipends for these institutes are all set at a uniform rate: $75 per week plus $15 per dependent. For detailed information concerning any particular institute, write to the director whose address is given below.

Carleton College, Northfield, Minnesota
June 16 to July 25

The Institute at Carleton College in Northfield, Minnesota, will be open to high school teachers of mathematics grades seven through twelve who have had at least three years of teaching experience. Participants will take two of the following five courses: The Number System, Fundamental Mathematics, Elementary Algebra from a Modern Point of View, Elementary Statistics, and Astronomy. The level of difficulty of these courses will be varied so as to meet the needs of teachers at different levels and with different educational backgrounds. Special lecturers will speak on Numerical Analysis, The History of Mathematics, Fundamental Ideas of the Calculus, Admission with Advanced Standing, High School Mathematics for Gifted Students, Topics from Topology, and Heuristic. Address inquiries to Dr. Kenneth O. May, Carleton College, Northfield, Minnesota.

Oberlin College, Oberlin, Ohio
June 16 to August 8

Courses:

1. Basic Concepts of Modern Mathematics, Professor R. R. Stoll. Set theory, mathematical logic, and various axiomatic systems.
2. Probability and Statistical Inference, Professor Samuel Goldberg. Elementary mathematical theory of probability: discrete and continuous variables. Application to testing statistical hypotheses.

In addition, Professor John Baum will present some recreational mathematics leading to fundamental notions. There will be some short series of lectures on such topics as the CEEB Commission on Mathematics, the University of Illinois Committee on School Mathematics, Game Theory and Linear Programming, Mathematics and the Social Sciences. Address inquiries to Dr. Wade Ellis, Oberlin College, Oberlin, Ohio.

Rutgers University, New Brunswick, New Jersey
June 29 to July 9 and June 29 to August 8

Two Mathematics Institutes will be conducted at Rutgers University.

1. June 29–July 9, discussion groups, lectures and laboratories designed to provide fuller understanding of subject matter and more effective ideas for the teaching of both elementary and high school mathematics. Credit optional.
2. June 29–August 8, a more intensive program designed particularly to acquaint high school teachers with some of the simpler, fundamental concepts of modern mathematics, and to assist them in preparing to teach such modernized curricula as that advocated by the Commission on Mathematics. For brochures and scholarship stipend application forms write Dr. Emory P. Starke, Director, Rutgers University, New Brunswick, New Jersey.

University of Buffalo, Buffalo 14, New York
July 7 to August 1

The Summer Institute at the University of Buffalo will offer four week-long courses in Modern Concepts of Mathematics by four prominent lecturers in the Historical Development of Mathematics, Algebra, Geometry, and some aspects of Applied Mathematics. An unusual feature of this institute will be the presence of twenty-five gifted eleventh-grade students who will have the opportunity of working on challenging problems with the visiting lecturers. They will also serve as a demonstration group to show how modern concepts in mathematics can be presented to students at their level. Address inquiries to Dr. Harriet F. Montague, University of Buffalo, Buffalo 14, New York.

University of Chicago, Chicago 37, Illinois
June 23 to August 1

The general purpose of the Summer Institute at the University of Chicago is to extend the mathematical background and professional competence of high school mathematics teachers. To this end a program of courses, seminars, lectures, and demonstrations has been especially designed for the participants.

The major part of this program deals with new perspectives on fundamental questions concerning geometry in school, and modern treatments of some topics in algebra and geometry. The remainder includes occasional lectures and a seminar devoted to content and teaching

of high school mathematics; systematic observation of demonstration classes of high school mathematics is planned. Address inquiries to Dr. A. L. Putnam, Box 23, Eckhart Hall, University of Chicago, Chicago 37, Illinois.

University of Kansas, Lawrence, Kansas
June 9 to August 2

The University of Kansas 1958 NSF Summer Institute for High School and College Teachers of Mathematics will offer four courses. A Demonstration Class of gifted high school students will be employed by Professor Beberman and his associates from Illinois in teaching the first course. Professor George Springer will teach a freshman college course dealing with sets, logic, probability, and other topics. One of two more advanced courses will probably treat mathematics for the social sciences. All courses will carry credit in the Graduate School. For information and application blanks, write to Dr. G. Baley Price, University of Kansas, Lawrence, Kansas.

University of Notre Dame, Notre Dame, Indiana
June 19 to August 6

Notre Dame Institute is a part of a multilevel program of mathematics teacher-training leading to an M.S. degree. Its aim is to introduce the teacher to those ideas of modern mathematics the mastery of which would enable him to meet effectively the exacting demands made by the changing high-school curriculum trends. The basic courses in Number Theory, Higher Algebra, Geometry, and Analysis and the advanced elective courses are taught by the regular staff as well as by distinguished visitors (Dr. G. Y. Rainich and Dr. H. D. Kloosterman in the summer of 1957). A detailed description of the program sent upon request. Address inquiries to Dr. Arnold E. Ross, University of Notre Dame, Notre Dame, Indiana.

University of Vermont, Burlington, Vermont
June 23 to August 8

A seven-week Mathematics Institute for High School teachers will be held at the University of Vermont next summer starting June 23. The titles of the three courses that will be offered are "Fundamental Concepts of Algebra," "Fundamental Concepts of Geometry," and "The Development of Mathematics." The objectives are:

1. to acquaint the participants with modern developments in algebra and geometry, and to have them study these subjects from an advanced point of view.
2. to inform the participants of the historical development of mathematics.
3. to relate, by means of discussion sessions, the above studies to each participant's own work.

Address inquiries to Dr. N. James Schoonmaker, University of Vermont, Burlington, Vermont.

University of Wisconsin, Madison 6, Wisconsin
June 30 to August 22

The Institute at the University of Wisconsin will be designed primarily for teachers of senior high school mathematics. Participants will take courses designed to give them command of the senior high school mathematics as recommended by the Commission on Mathematics of the College Entrance Examination Board. Two mathematics courses will be offered: Statistics for High School Teachers and Topics in Elementary Analysis. A seminar on the High School Mathematics Curriculum will be offered. Address inquiries to Dr. H. Van Engen, University of Wisconsin, Madison 6, Wisconsin.

University of Wyoming, Laramie, Wyoming
June 16 to August 8

The University of Wyoming offers a Summer Mathematics Institute for high school teachers. All participants will take a course in "Foundations of Mathematics," exploring such topics as theory of sets, the axiomatic method, and symbolic logic. For their second course, participants would take one of two specially planned courses: Calculus or an introductory course in Real Variables. Additional occasional lectures will be given.

All high school teachers of mathematics with at least an undergraduate minor in the subject are eligible for stipends. Address inquiries to Dr. W. N. Smith, Mathematics Summer Institute, University of Wyoming, Laramie, Wyoming.

Western Michigan University, Kalamazoo, Michigan
June 23 to August 1

A major objective of this Institute will be to emphasize some contemporary concepts of mathematics, to acquaint teachers with relevant literature, and to assist them in the study of it. The concept of number, considered from a variety of viewpoints, will provide a focus for a general "course" to be taken by all participants. Supplementing this central course will be (a) four smaller "interest groups" and (b) a series of guest lectures. Any person now teaching high school mathematics, whose mathematical training includes calculus and who holds a bachelor's degree, may apply. Stipends for forty-five participants will be available. Address inquiries to Dr. Charles H. Butler, Western Michigan University, Kalamazoo, Michigan.

Summer institutes for science and mathematics teachers

The National Science Foundation is sponsoring institutes for science and mathematics teachers at the schools listed below. For information as to the branch of science emphasized and the mathematics courses offered at each institute write to the director whose name is given in parentheses.

University of Alabama, University, Alabama (Charles K. Arey, Department of Education)

University of Alaska, College, Alaska (Alfred M. Bork, Department of Physics)

University of Arizona, Tucson, Arizona (Millard G. Seeley, Department of Chemistry)

University of Arkansas, Fayetteville, Arkansas (Lowell F. Bailey, Department of Botany)

Colorado College, Colorado Springs, Colorado (Lewis N. Pino, Cutler Hall)

Wesleyan University, Middletown, Connecticut (Vincent W. Cochrane, Department of Biology)

University of Delaware, Newark, Delaware (Cecil C. Lynch, Department of Chemistry)

Florida State University, Tallahassee, Florida (J. Paul Reynolds, College of Arts and Sciences)

Atlanta University, Atlanta, Georgia (K. A. Huggins, Department of Chemistry)

University of Georgia, Athens, Georgia (T. H. Whitehead, Department of Chemistry)

University of Hawaii, Honolulu, Hawaii (John J. Naughton, Department of Chemistry)

University of Idaho, Moscow, Idaho (W. H. Cone, Department of Physical Sciences)

Illinois Wesleyan University, Bloomington, Illinois (Wayne W. Wantland, Science Division)

Iowa State College, Ames, Iowa (J. A. Greenlee, Science Division)

Louisiana State University, University Station, Baton Rouge, Louisiana (H. B. Williams, Department of Chemistry)

Colby College, Waterville, Maine (Evans B. Reid, Department of Chemistry)

Morgan State College, Baltimore 12, Maryland (Thomas P. Fraser, Science Education)

Tufts University, Medford, Massachusetts (M. Kent Wilson, Department of Chemistry)

Michigan State University, East Lansing, Michigan (F. B. Dutton, Science and Mathematics Teaching Center)

Northern Michigan College, Marquette, Michigan (Holmes Boynton, Department of Mathematics)

University of Minnesota (Duluth), Duluth, Minnesota (W. R. McEwen, Science and Mathematics Division)

University of Mississippi, University, Mississippi (George Vaughan, Department of Chemistry)

Missouri School of Mines and Metallurgy, Rolla, Missouri (Harold Q. Fuller, Department of Physics)

University of Nebraska, Lincoln 8, Nebraska (Walter E. Militzer, College of Arts and Sciences)

Union College, Schenectady, New York (George Reed, Department of Chemistry)

Duke University, Durham, North Carolina (W. M. Nielsen, Department of Physics)

North Carolina College at Durham, Durham, North Carolina (William H. Robinson, Department of Physics)

University of North Carolina, Chapel Hill, North Carolina (Victor A. Greulach, Department of Botany)

University of North Dakota, Grand Forks, North Dakota (J. Donald Henderson, Department of Physics)

Ohio University, Athens, Ohio (L. P. Eblin, Department of Chemistry)

Allegheny College, Meadville, Pennsylvania (Robert E. Bugbee, Department of Biology)

Bucknell University, Lewisburg, Pennsylvania (Lester Kieft, Department of Chemistry)

University of Puerto Rico, Rio Piedras, Puerto Rico (F. Bueso, College of Natural Sciences)

Brown University, Providence 12, Rhode Island (Elmer R. Smith, Teacher Training)

State University of South Dakota, Vermillion, South Dakota (Charles R. Estee, Box 57, University Exchange)

Tennessee Polytechnic Institute, Cookeville, Tennessee (G. B. Pennebaker, School of Arts and Sciences)

Baylor University, Waco, Texas (Bryce C. Brown, Department of Biology)

Howard Payne College, Brownwood, Texas (Richard A. Eads, Science Division)

Southern Methodist University, Dallas, Texas (Joe P. Harris, Jr., Department of Biology)

Stephen F. Austin State College, Nacogdoches, Texas (H. E. Abbott, Department of Chemistry)

State College of Washington, Pullman, Washington (Alfred B. Butler, Department of Physics)

West Virginia University, Morgantown, West Virginia (J. K. Stewart, Department of Mathematics)

Ripon College, Ripon, Wisconsin (Robert S. Wilson, Department of Physics)

Academic Year Institutes for secondary mathematics teachers

Two Academic Year Institutes are being sponsored by the National Science Foundation for teachers of secondary mathematics. For detailed information write to the directors.

The general purpose of the Academic Year Institute at the University of Chicago is to extend the mathematical background and professional competence of high school mathematics teachers. To this end a program of courses, seminars, lectures, and demonstrations has been especially designed for the participants.

The major part of this program consists of courses presenting topics from modern algebra, geometry, analysis, and statistics. The remaining part includes a seminar devoted to the content and teaching of high school mathematics; the seminar serves also as a focus for systematic observation of a college class in basic mathematics, and for occasional special lectures. For information write to Dr. E. P. Northrop, Box 23, Eckhart Hall, University of Chicago, Chicago 37, Illinois.

The purposes of the Academic Year Institute (1958–59) at the University of Illinois are: to enrich the teachers' mathematical backgrounds by acquainting them with some modern developments in mathematics; to inform them of new curriculum developments; and to provide them with specific techniques in pedagogy.

The mathematical subject matter includes: logic, theory of sets, geometry, modern algebra, elementary theory of functions, statistics, and several optional topics. Two courses in pedagogy will be offered in collaboration with the University of Illinois Committee on School Mathematics Project. There will be a course in topics on the philosophy of education and learning theory.

For information, address Dr. Joseph Landin, 303 Altgeld Hall, University of Illinois, Urbana, Illinois.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

JOINT MEETING WITH NEA

June 30, 1958
Cleveland, Ohio
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

EIGHTEENTH SUMMER MEETING

August 19–20, 1958
Colorado State College of Education, Greeley, Colorado
Forest N. Fisch, Colorado State College of Education, Greeley, Colorado

CHRISTMAS MEETING

December 29–30, 1958
Sheraton-McAlpin Hotel, New York, New York
Abraham S. Kadish, High School of Fashion Industries, 225 West 24th Street, New York 11, New York

Other professional dates

Sectional Meetings of the Illinois Council of Teachers of Mathematics

March 15, 1958, Belleville Township High School, Belleville, Illinois

March 22, 1958, Hinsdale Township High School, Hinsdale, Illinois and Western Illinois University, Macomb, Illinois

April 16, 1958, Eastern Illinois University, Charleston, Illinois

April 19, 1958, Southern Illinois University, Carbondale, Illinois and Illinois State Normal University, Normal, Illinois

Francis R. Brown, Illinois State Normal University, Normal, Illinois

For information about the dates of the summer and academic year institutes of the National Science Foundation, see pages 149–152.

Letters to the editor

My dear Mr. Editor:

Will you please enlighten your readers *re* certain points not mentioned in Rev. Brother Leo's article—"A Mental Calendar," in the October 1957 issue.

Professor Isaac Krieger and writer were the first to publicize a system similar to one mentioned by Brother Leo, during Krieger's lectures at McGill University, Montreal, Canada, in 1928 and 1929 before 500 scientists. However, our system was enlarged to start from year A.D. 1, if necessary, allowing for change during year 1752.

Fred Barlow in his book *Mental Prodigies* (Philosophical Library) also endeavored to give the impression that he was one of the first to discover the calendar stunt. I wrote Fred to advise him otherwise. Krieger and I addressed the University (McGill) on lightning computations with logarithms and we demonstrated certain magic squares.

However, Brother Leo deserves to be complimented on bringing the mental calendar to your attention, and thereby teaching so many qualified readers.

Since 1929 there have been over 15 different articles similar to the one written by Brother Leo.

Yours sincerely,
H. V. Gosling, M.A., Ph.D.
27 Hillendale Ave.
Kingston, Ontario, Canada

P. S. What does your reviewer mean when he refers to the Dale procedure (page 457, October issue)? O. F. Schaaf, reviewer.

Editorial Note: Dr. Schaaf was referring to the Edgar Dale reading scale.

To the Editor of The Mathematics Teacher:

I find that in some preparatory schools the teachers are having their pupils use a pair of dots to enclose binomials, etc., instead of a pair of parentheses. Can anything be done to check this? We write

$$a \cdot b + c \cdot \sqrt{x} \text{ to mean } ab + c\sqrt{x}.$$

Shall we allow an ambiguous notation that may also mean $(ab + ac)\sqrt{x}$? Has anyone a defense for this practice?

William R. Ransom,
State College,
Dover, Delaware

Dear Editor:

As a citizen of another American nation and one who is concerned about the accurate use of statistics—particularly in a magazine with the word "mathematics" in its title—I should like to reply to the challenge presented by Mr. Doremus in the October 1957 issue of The Mathematics Teacher. His article, which purports to refute what he considers to be "misleading, unreliable, and frequently outdated" evidence that "American youngsters are inferior in every way to European-schooled youngsters," is a distressing example of what appears to be emotional bias combined with misuse of data that are probably in themselves objective, and reliable.

In the first place, the comparisons he is concerned about generally appear unfavorable to American children chiefly because of the highly selective type of education in most European countries. It is simply unreasonable to compare students in an American—or Canadian—high school with those in a British grammar school, a French lycée, or a German gymnasium. Even the college-bound students in our high schools are not as highly selected as those in the types of schools I have mentioned. In the elementary schools also, where there is less selection everywhere, one needs to interpret with caution any

differences which are found. There is, for example, nothing in the United States or Canada which can compare with the British "eleven-plus" examination—controversial even in Britain—and this examination undoubtedly affects the curriculum and the teaching emphasis in classes which are approximately at our fifth- or sixth-grade level.

My own feeling is that our basic philosophy of equal concern for *all* pupils is the better one, and that the British, for example, do not do as good a job as we do with the average and slower pupils. I am willing to admit that they—like the Russians—make better provisions for the more able students, but I would not concede that we cannot do as well in our present framework.

In the second part of his article Mr. Doremus reports and interprets certain data in a way which distresses me even more than the point of view which he expresses earlier, for the following reasons:

1. From its title, the British *Cotswald Measurement of Ability Test* does not seem to be intended mainly as a measure of arithmetic achievement. According to the *Fourth Mental Measurements Yearbook* the test is "part of a battery designed for use with British children of eleven years of age who are passing from primary school to secondary school education. . . . The tests are in wide use in England for selecting children who are deemed suitable for secondary grammar school education." However, the content does appear to be mainly arithmetical, since it is stated later in the same review that "the testing material is so arranged as to provide balanced testing of most of the 390 basic arithmetical combinations, the simplest of measures in common use, and the reading and reasoning involved in the solution of simple problems." In a report on the use of a test likely to be unfamiliar to the reader, some description of the content seems to be indicated, so that the reader can make some judgment about the validity of the test for its present use. Mr. Doremus tells us only that it "has gained wide acceptance abroad as a valid measuring instrument in the arithmetic area."
2. I assume that the "medium (*sic*) standardized score obtained by British children" was suitably chosen, but the reader is not sufficiently informed about this matter. Because the organization of British schools is so different from ours, it seems particularly necessary to explain the basis of comparison in a case like this.
3. It is most inappropriate to compare 100 Glen Rock sixth-graders with norms developed in "general use through the schools of the United Kingdom" and then to draw conclusions about differences between American- and European-schooled children. We are given no evidence that Glen Rock sixth-graders can properly be considered to be typical American children. Assuming that the British results were based on an appropriate and unselected population, I would hazard a guess that Glen Rock is not a typical American community in several ways. It is probably true that the average IQ of its elementary school children—or at least of the children who were given the British test—is significantly above 100, that a higher proportion of its children go to college than in the state as a whole, and that the community contains a higher proportion of professional and white-collar workers than one would find in the country generally. It seems to me, on the basis of the scanty evidence given, that Mr. Doremus has shown that a fairly large proportion of Glen Rock children would be admitted to a British grammar school, and this means that they are considerably above average in academic aptitude, which everyone in Glen Rock probably knows anyway.

I would conclude this letter with a plea for a less defensive attitude about our educational product, for a more realistic appraisal of both our philosophy and our methods, and most of all, for accurate and clear reporting of results from investigations such as this, modest though they be admitted to be.

I might add that in the course of the testing program for students entering the Institute of Education we have discovered some interesting differences between the scores on the quantitative part of the *College Ability Test* obtained by our students and those of the American college freshmen represented by the test norms. I hope to complete a brief report on this matter in the near future; perhaps it would be of interest to the readers of THE MATHEMATICS TEACHER.

Yours very truly,
(Miss) FRANCES E. CROOK
Assistant Professor of Education
Macdonald College of McGill University
Quebec, Canada

Dear Sir,

We thought the readers of THE MATHEMATICS TEACHER would be interested in the following formula we have worked out which may be used to find the t^{th} term of any binomial expansion of the form $(a \pm b)^n$. It might be noted that this formula works equally well with fractions in place of a and b.

$$t^{\text{th}} \text{ term} = \frac{P_{n,(t-1)}}{(t-1)!}\, a^{n-(t-1)} b^{(t-1)}$$

from which we get

$$t^{\text{th}} \text{ term} = C_{n,(t-1)} a^{n-(t-1)} b^{(t-1)}.$$

Sincerely yours,
BOB DEIHL
SAM WILLIAMSON
RUSSELL CHAFFEE,
Sayre Area Joint High School
Sayre, Pennsylvania

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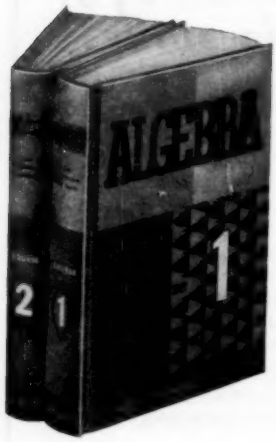

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